New approach for the study of linear Vlasov stability of inhomogeneous systems

Enrico Camporeale^{a)}

Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, United Kingdom

Gian Luca Delzanno^{b)} and Giovanni Lapenta^{c)}

T-15 Plasma Theory Group, Los Alamos National Laboratory, 87545 Los Alamos, New Mexico 87545

William Daughton^{d)}

Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242

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This paper presents an alternative technique for solving the linearized Vlasov-Maxwell set of equations, in which the velocity dependence of the perturbed distribution function is described by means of an infinite series of orthogonal functions, chosen as Hermite polynomials. The orthogonality properties of such functions allow us to decompose the Vlasov equation into a set of infinite coupled linear equations. With a suitable truncation relation, the problem is transformed in an eigenvalue problem. This technique is based on solid but easy concepts, not attempting to evaluate the integration over the unperturbed trajectories and can be applied to any equilibrium. Although the solutions are approximate, because they neglect contributions of higher order coefficients of the series, the physical meaning of the low-order coefficients is clear. Furthermore the accuracy of the solution, which depends on the number of terms taken into account in the Hermite series, appears to be merely a problem of computational power. The method has been tested for a 1D Harris equilibrium, known to give rise to several instabilities like tearing, drift kink, and lower hybrid. The results are shown in agreement with those obtained by Daughton with a traditional technique based on the integration over unperturbed orbits. © 2006 American Institute of Physics. [DOI: 10.1063/1.2345358]

I. INTRODUCTION

The issue of the linear stability of the Vlasov-Maxwell system has been deeply investigated for the past 40 years. The first complete mathematical theory for solving the linearized Vlasov equation, for the case of a field-free plasma, is due to Landau,¹ that introduced the well-known technique of Landau contours in order to invert the Laplace transform of the perturbed distribution functions. The need for a more general theory led to investigate the so-called method of characteristics.² It consists of integrating the Vlasov equation in time from $-\infty$ to t along a path in phase space which coincides with the orbit of the charged particles in the equilibrium electric and magnetic fields. The problem can be easily solved if some approximations are made, and several authors used this approach applied to the configuration of a magnetic neutral sheet. For instance in Ref. 3 a simplified model in which the particle orbits were considered as straight lines was introduced. The same strategy has been widely used in a number of subsequent works (see, for example, Refs. 4 and 5). In Ref. 6 only symmetrical modes with respect to the magnetic neutral line were considered, while electrostatic perturbations were neglected.

An algorithm where the evaluation of the exact orbit

integrals was performed using the particle-in-cell (PIC) simulation was devised by Brittnacher *et al.*,⁷ while an exact numerical evaluation of the orbit integral was introduced by Daughton^{8–10} and followed by Sitnov *et al.*¹¹ Much effort has been done in conducting PIC code simulations for investigating both the linear and nonlinear behavior of such configurations (see, for example, Refs. 12–15).

In this paper, we present an alternative approach based on a generalization of the method presented in Refs. 16 and 17. The mathematical technique is well-known and widely used in many fields of physics, nevertheless, to the best of our knowledge, it has not been applied to the study of the Vlasov-Maxwell linear stability. The main idea consists of expanding the distribution function in a series of Hermite polynomials in velocity space. With this procedure, one obtains an infinite set of equations, which must be truncated by means of a closure relation. It is clear that this approach stands on much easier concepts than all previous work, because it avoids the issue of integrating over unperturbed orbits in phase space, while the various terms of expansion have a clear physical meaning. The novelty introduced by the new approach is threefold.

First, the use of Hermite polynomial expansion in solving a linear stability problem is, to the knowledge of the authors, new. Hermite polynomials have been used extensively in deriving fluid models from kinetic models, but not as a general tool to analyze the linear stability of kinetic

^{a)}Electronic mail: e.camporeale@qmul.ac.uk

^{b)}Electronic mail: delzanno@lanl.gov

^{c)}Electronic mail: lapenta@lanl.gov

^{d)}Electronic mail: william-daughton@uiowa.edu

models. The advantages of the new approach are both practical and theoretical. On the practical side, the approach proposed here is relatively simple and straightforward from the algebraic point of view. Its implementation in a computer code does not require advanced numerical methods or powerful parallel computers. The results shown here are obtained with a relatively short program written in MATLAB and runs on a single processor machine.

Second, the use of Hermite polynomial expansion has the great advantage that each term in the expansion corresponds to a moment of the distribution function with a precise physical meaning. This feature allows us to investigate directly important higher order moments of the distribution function such as the pressure tensor. Previous methods could of course provide the same information but at the cost of additional work and postprocessing of the results of linear stability analysis. Our approach, instead, provides high order moments naturally and at no additional cost.

Third, previous approaches to linear stability necessitated an equilibrium solution in closed form. Here, instead, the method based on Hermite polynomials only needs the reference equilibrium state in terms of its moments (terms in the Hermite series expansion). This advantage is very significant since in many practical cases an exact solution of the Vlasov equation is not known or it is only known numerically. The possibility of describing it via moments of the distribution (such as density, current, and pressure) makes our approach a much more practical stability analysis tool for practical cases.

The paper is organized as follows: the mathematical method is discussed in Sec. II. In Sec. III a technique to improve the convergence of the method is presented. Our approach has been tested for a simple Harris current sheet equilibrium, and the results are presented in Sec. IV, where drift kink (DKI), lower hybrid drift (LHDI), and tearing instabilities are examined. Conclusions are drawn in Sec. V.

II. MATHEMATICAL METHOD

We consider a collisionless plasma of electrons and singly charged ions, described by the well-known Vlasov-Maxwell system of equations. We choose a Cartesian reference system (x, y, z) with cgs units.

All quantities are normalized according to the following choices: lengths are referred to the characteristic length L of the equilibrium (in this paper we will focus on Harris-type equilibria, so that L is the current sheet thickness); velocities are referred to the speed of light c; magnetic and electric fields are referred to an equilibrium field B_0 , while densities are normalized on a reference density n_0 .

In order to study the linear stability of the system, we linearize Vlasov's equation

$$\frac{\partial f_{1s}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_{1s}}{\partial \boldsymbol{r}} + \Omega_{cs}(\boldsymbol{E}_0 + \boldsymbol{v} \times \boldsymbol{B}_0) \cdot \frac{\partial f_{1s}}{\partial \boldsymbol{v}}$$
$$= -\Omega_{cs}(\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_1) \cdot \frac{\partial f_{0s}}{\partial \boldsymbol{v}}, \qquad (1)$$

where f is the distribution function, E and B are electric and

magnetic fields, and initial and perturbed quantities are denoted, respectively, by the subscripts 0 and 1. The subscript *s* labels the species (s=e,i), $\Omega_{cs}=q_sB_0L/(m_sc^2)$ is the dimensionless gyrofrequency for the species *s*, m_s , and q_s are the mass and the charge of the plasma particles, respectively. Equation (1) is coupled to the Maxwell equations for the perturbed fields. It is useful to introduce the scalar and vector potentials ϕ and A, which are related to the electric and magnetic fields through

$$\boldsymbol{E}_1 = -\nabla \boldsymbol{\phi}_1 - \frac{\partial \boldsymbol{A}_1}{\partial t},\tag{2}$$

$$\boldsymbol{B}_1 = \nabla \times \boldsymbol{A}_1. \tag{3}$$

Furthermore, we choose the Lorenz gauge:

$$\nabla \cdot \boldsymbol{A}_1 + \frac{\partial \phi_1}{\partial t} = 0 \tag{4}$$

so that Maxwell's equations read:

$$\nabla^2 \phi_1 - \frac{\partial^2 \phi_1}{\partial t^2} = -\sum_s \frac{\omega_{ps}^2}{\Omega_{cs}} \rho_{1s},\tag{5}$$

$$\nabla^2 A_1 - \frac{\partial^2 A_1}{\partial t^2} = -\sum_s \frac{\omega_{ps}^2}{\Omega_{cs}} j_{1s}, \qquad (6)$$

where $\omega_{ps} = (4 \pi n_0 q_s^2 L^2 / m_s c^2)^{1/2}$ is the plasma frequency calculated from the reference density n_0 . In Eqs. (5) and (6), the perturbed current and charge densities are determined from the solution of Eq. (1):

$$\rho_1 = \sum_s q_s \int f_{1s} d\boldsymbol{v},\tag{7}$$

$$\boldsymbol{j}_1 = \sum_{s} q_s \int \boldsymbol{v} f_{1s} d\boldsymbol{v} \,. \tag{8}$$

We choose perturbations of the following form:

$$\phi_1(\mathbf{x},t) = \phi(x)e^{\omega t}e^{ik_y y + ik_z z},\tag{9}$$

$$A_1(\mathbf{x},t) = A(x)e^{\omega t}e^{ik_y y + ik_z z}$$
(10)

while we assume the perturbed distribution function $f_{1s}(\boldsymbol{x}, \boldsymbol{v}, t)$ as an infinite series of Hermite polynomials in velocity space, weighted by a decreasing exponential function:

$$f_{1s}(\mathbf{x}, \mathbf{v}, t) = \frac{1}{\pi^{3/2}} \sum_{n, m, p} C_{n, m, p}^{s}(x, t) e^{ik_{y}y} e^{ik_{z}z} e^{-(\xi_{x}^{2} + \xi_{y}^{2} + \xi_{z}^{2})} \\ \times H_{n}(\xi_{x}) H_{m}(\xi_{y}) H_{p}(\xi_{z}), \qquad (11)$$

where $C_{n,m,p}^{s}$ is function of (x,t), $\xi_{x}=(v_{x}-u_{s})/\alpha_{s}$, $\xi_{y}=(v_{y}-v_{s})/\beta_{s}$, $\xi_{z}=(v_{z}-w_{s})/\gamma_{s}$ and H_{n} is the *n*th order Hermite polynomial. The velocities u_{s} , v_{s} , w_{s} , and α_{s} , β_{s} , γ_{s} are assumed constant in this paper [although in Appendix B the reader can find the generalization of expansion (11) to the case where u_{s} , v_{s} , w_{s} , and α_{s} , β_{s} , γ_{s} are functions of x].

We notice that this specific choice of the weighting function was first considered by Grad in Ref. 18. It is particularly

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useful to allow the function f_{1s} to satisfy the condition $f_{1s} \rightarrow 0$ when $v \rightarrow \infty$. Furthermore, it is the more natural choice due to the orthogonality property of Hermite polynomials. The product of Hermite polynomials, each of them a function of a single variable, is not of course the more general choice, and one could argue about the completeness and the orthogonality properties of such a base. Nevertheless our choice is a special case of the definition of general Hermite two-dimensional polynomials given by Wünsche¹⁹ (trivially

extended to 3D), for which completeness and orthogonality properties have been proven.

By substituting definition (11) in Eq. (1), after multiplication by $H_q(\xi_x)H_r(\xi_y)H_t(\xi_z)$ and integration in velocity space, and by means of the orthogonality properties of the Hermite polynomials, the problem can be decomposed in a set of infinite equations for the coefficients $C_{q,r,t}^s(x,t)$ $(q,r,t=0,...,\infty)$, each of them involving both the perturbed potentials ϕ_1 and A_1 , and coefficients $C_{q,r,t}$ of different order:

$$\frac{\partial C_{q,r,t}^{s}}{\partial t} = -\alpha_{s} \left[\frac{\partial C_{q+1,r,t}^{s}}{\partial x} (q+1) + \frac{1}{2} \frac{\partial C_{q-1,r,t}^{s}}{\partial x} + \frac{u_{s}}{\alpha_{s}} \frac{\partial C_{q,r,t}^{s}}{\partial x} \right] - \beta_{s} i k_{y} \left[(r+1) C_{q,r+1,t}^{s} + \frac{1}{2} C_{q,r-1,t}^{s} + \frac{v_{s}}{\beta_{s}} C_{q,r,t}^{s} \right] \\ - \gamma_{s} i k_{z} \left[(t+1) C_{q,r,t+1}^{s} + \frac{1}{2} C_{q,r,t-1}^{s} + \frac{w_{s}}{\gamma_{s}} C_{q,r,t}^{s} \right] + \frac{\Omega_{cs}}{\alpha_{s}} E_{x0} C_{q-1,r,t}^{s} + \frac{\Omega_{cs}}{\beta_{s}} E_{y0} C_{q,r-1,t}^{s} + \frac{\Omega_{cs}}{\gamma_{s}} E_{z0} C_{q,r,t-1}^{s} \right] \\ - \Omega_{cs} B_{x0} \left[\frac{\beta_{s}}{\gamma_{s}} C_{q,r+1,t-1}^{s} (r+1) - \frac{\gamma_{s}}{\beta_{s}} C_{q,r-1,t+1}^{s} (t+1) + \left(\frac{\beta_{s}}{\gamma_{s}} - \frac{\gamma_{s}}{\beta_{s}} \right) \frac{1}{2} C_{q,r-1,t-1}^{s} + \frac{v_{s}}{\gamma_{s}} C_{q,r,t-1}^{s} - \frac{w_{s}}{\beta_{s}} C_{q,r-1,t}^{s} \right] \\ - \Omega_{cs} B_{y0} \left[\frac{\gamma_{s}}{\alpha_{s}} C_{q-1,r,t+1}^{s} (t+1) - \frac{\alpha_{s}}{\gamma_{s}} C_{q+1,r,t-1}^{s} (q+1) + \left(\frac{\gamma_{s}}{\alpha_{s}} - \frac{\alpha_{s}}{\gamma_{s}} \right) \frac{1}{2} C_{q-1,r,t-1}^{s} + \frac{w_{s}}{\alpha_{s}} C_{q-1,r,t}^{s} - \frac{w_{s}}{\gamma_{s}} C_{q,r,t-1}^{s} \right] \\ - \Omega_{cs} B_{z0} \left[\frac{\alpha_{s}}{\beta_{s}} C_{q+1,r-1,t}^{s} (q+1) - \frac{\beta_{s}}{\alpha_{s}} C_{q-1,r+1,t}^{s} (r+1) + \left(\frac{\alpha_{s}}{\beta_{s}} - \frac{\beta_{s}}{\alpha_{s}} \right) \frac{1}{2} C_{q-1,r,t-1}^{s} + \frac{w_{s}}{\beta_{s}} C_{q,r,t-1}^{s} - \frac{w_{s}}{\gamma_{s}} C_{q,r,t-1}^{s} \right] \\ - \frac{\Omega_{cs} B_{z0}} \left[\frac{\alpha_{s}}{\beta_{s}} C_{q+1,r-1,t}^{s} (q+1) - \frac{\beta_{s}}{\alpha_{s}} C_{q-1,r+1,t}^{s} (r+1) + \left(\frac{\alpha_{s}}{\beta_{s}} - \frac{\beta_{s}}{\alpha_{s}} \right) \frac{1}{2} C_{q-1,r-1,t}^{s} + \frac{w_{s}}{\beta_{s}} C_{q,r-1,t}^{s} - \frac{w_{s}}{\alpha_{s}} C_{q-1,r,t}^{s} \right] \\ - \frac{\Omega_{cs}}{2^{q+r+t}} \frac{1}{q! r! t!} \int (E_{1} + \mathbf{v} \times B_{1}) \cdot \frac{\partial f_{0}}{\partial \mathbf{v}} H_{q}(\xi_{s}) H_{r}(\xi_{s}) H_{t}(\xi_{s}) d\xi, \qquad (12)$$

where the dependence upon x and t has been omitted, and $d\xi = d\xi_x d\xi_y d\xi_z$.

Maxwell's equation (5), after substitution of Eq. (11), reads as

$$\frac{\partial^2 A_{x1}}{\partial t^2} = \frac{\partial^2 A_{x1}}{\partial x^2} - (k_y^2 + k_z^2) A_{x1} + \frac{\alpha_i \beta_i \gamma_i \Omega_{ci}}{v_{\text{th},i}^2 (1 + T_e/T_i)} \\ \times (\alpha_i C_{1,0,0}^i + u_i C_{0,0,0}^i) + \frac{\alpha_e \beta_e \gamma_e \Omega_{ce}}{v_{\text{th},e}^2 (1 + T_i/T_e)} \\ \times (\alpha_e C_{1,0,0}^e + u_e C_{0,0,0}^e), \qquad (13)$$

$$\frac{\partial^2 A_{y1}}{\partial t^2} = \frac{\partial^2 A_{y1}}{\partial x^2} - (k_y^2 + k_z^2) A_{y1} + \frac{\alpha_i \beta_i \gamma_i \Omega_{ci}}{v_{\text{th},i}^2 (1 + T_e/T_i)} \\ \times (\beta_i C_{0,1,0}^i + v_i C_{0,0,0}^i) + \frac{\alpha_e \beta_e \gamma_e \Omega_{ce}}{v_{\text{th},e}^2 (1 + T_i/T_e)} \\ \times (\beta_e C_{0,1,0}^e + v_e C_{0,0,0}^e), \qquad (14)$$

$$\begin{aligned} \frac{\partial^2 A_{z1}}{\partial t^2} &= \frac{\partial^2 A_{z1}}{\partial x^2} - (k_y^2 + k_z^2) A_{z1} + \frac{\alpha_i \beta_i \gamma_i \Omega_{ci}}{v_{\text{th},i}^2 (1 + T_e/T_i)} \\ &\times (\gamma_i C_{0,0,1}^i + w_i C_{0,0,0}^i) + \frac{\alpha_e \beta_e \gamma_e \Omega_{ce}}{v_{\text{th},e}^2 (1 + T_i/T_e)} \\ &\times (\gamma_e C_{0,0,1}^e + w_e C_{0,0,0}^e), \end{aligned}$$
(15)

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} - (k_y^2 + k_z^2)\phi + \frac{\alpha_i \beta_i \gamma_i \Omega_{ci}}{v_{\text{th},i}^2 (1 + T_e/T_i)} C_{0,0,0}^i$$
$$+ \frac{\alpha_e \beta_e \gamma_e \Omega_{ce}}{v_{\text{th},e}^2 (1 + T_i/T_e)} C_{0,0,0}^e, \tag{16}$$

where $v_{\text{th},s} = \sqrt{2T_s/m_sc^2}$ is the dimensionless thermal velocity, T_s is the temperature of the species *s*, and the identity $B_0^2 = 8\pi n_0(T_e + T_i)$ has been used. Finally, the linear stability of the system is studied as an eigenvalue problem, by assuming the time dependence of coefficients $C_{n,m,p}$ of the form $e^{\omega t}$ and splitting each of the second order Maxwell equation (13)–(16) in two first order equations. Equation (12) together with Maxwell's equations can thus be written in the form:

$$\omega \mathbf{X} = \hat{M} \mathbf{X},\tag{17}$$

where \hat{M} is the coefficient matrix and the vector **X** is defined as

$$\mathbf{X} = \left(C_{0,0,0}^{i}, C_{0,0,1}^{i}, \dots, C_{0,0,0}^{e}, C_{0,0,1}^{e}, \dots, \frac{\partial A_{x1}}{\partial t}, \frac{\partial A_{y1}}{\partial t}, \frac{\partial A_{z1}}{\partial t}, \frac{\partial \phi_{1}}{\partial t}, A_{x1}, A_{y1}, A_{z1}, \phi_{1}\right)^{T}.$$

A Dirichlet-type boundary condition is applied for all the coefficients $C_{q,r,t}^s$ and the following potentials: A_{y1} , A_{z1} , ϕ_1 ,

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while a Neumann-type condition is applied to $A_{x1} (dA_{x1}/dx = 0)$. With this particular choice, the Lorenz gauge is satisfied exactly at the boundary.

The eigenvalues are, in general, complex numbers $\omega = \omega_r + i\omega_i$ in which the real part determines the growth or the damping of the linear mode. Clearly any mode for which $\omega_r > 0$ is unstable.

It is important to emphasize that, in the framework of this method, low order coefficients have a well-known physical meaning due to their relation to the momenta of the perturbed distribution function. In fact, recalling the definition and orthogonality property of the Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n (e^{-x^2})}{dx^n},$$
(18)

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \delta_{n,m} 2^n n! \sqrt{\pi}$$
(19)

one gets easily, from Eq. (11):

$$C_{0,0,0}^{s} = \frac{n_{s}}{v_{\text{th},s}^{3}},\tag{20}$$

$$C_{1,0,0}^{s} = \frac{n_{s} V_{x,s}}{v_{\text{th},s}^{4}} - \frac{u_{s}}{v_{\text{th},s}} C_{0,0,0}^{s},$$
(21)

$$C_{2,0,0}^{s} = \frac{p_{xx,s}}{2v_{\text{th},s}^{5}} - \frac{1}{4}C_{0,0,0}^{s},$$
(22)

$$C_{1,1,0}^{s} = \frac{p_{xy,s}}{v_{\text{th},s}^{5}},$$
(23)

$$C_{3,0,0}^{s} = \frac{Q_{xx,s}}{3n_{s}m_{s}v_{\text{th},s}^{6}} - \frac{1}{4}C_{1,0,0}^{s}, \qquad (24)$$

where n_s is the number density, $V_{x,s}$ is the mean velocity along the x direction, $p_{xx,s}$ and $p_{xy,s}$ are, respectively, the components of the pressure tensor along the directions xxand xy, and $Q_{xx,s}$ is a term proportional to the flux of kinetic energy for particles of species s crossing a unit area perpendicular to the x direction.

Note that Eqs. (20)–(24) are valid when the free parameters u_s , v_s , w_s , and α_s , β_s , γ_s are equal, respectively, to the drift and thermal velocities of the equilibrium distribution functions.

Obviously some closure relation is needed for solving the equations. Let us now look closely at the physical meaning of some possible closure relations, following Ref. 16. Setting $C_{n,m,p}=0$ for $n+m+p \ge 2$ implies a scalar pressure $(p=p_{xx}=p_{yy}=p_{zz})$ which is proportional to the density, similarly to the law of a perfect gas:

$$p_s = \frac{1}{2} n_s v_{\text{th},s}^2.$$
 (25)

Further on, setting $C_{n,m,p}=0$ for $n+m+p \ge 3$ implies a nonvanishing heat flux $Q_{j,s}(j=x,y,z)$ only when the average velocity $V_{j,s}$ is not zero:

$$Q_{j,s} = \frac{3}{2} n_s^2 T_s V_{j,s}.$$
 (26)

The typical choice is to set to zero all coefficients $C_{n,m,p}$ for which n+m+p is greater than a fixed number N. All results presented in Sec. IV have been obtained with this choice, that has the physical meaning that the distribution functions is not allowed to have more than a certain number of maxima or minima. Of course, if N is large enough, this choice does not affect the result. An alternative choice would be to fix some integers (N, M, P) and assume that all coefficients $C_{n,m,p}$ with (n > N, m > M, p > P) vanish.

Depending on the physical problem studied, each of these two choices can be more efficient, that is the lowest number of coefficients is needed in order to achieve a good approximation of the function f_{1s} .

The advantages of this approach emerge clearly when compared to the methods traditionally used to address the problem of linear kinetic stability. In fact, we do not need to evaluate integrals along the unperturbed particle orbits, which is not an easy task unless some simplifications are introduced (without a clear physical meaning). With our method, one can take into account different numbers of terms in the expansions (11) and compare the solutions so obtained. The completeness of the set of Hermite polynomials guarantees the convergence of the solution, and one can theoretically achieve any precision just adding more terms in the expansion. Of course this is true within the limits posed by the computational resources available, and the number of terms needed to reach a reasonable accuracy depends on the physical problem studied. Nevertheless, we will show that at least for the particular problems used for benchmarking, an accurate solution can be reached. Furthermore, in the next section we will introduce a method that enables to improve the convergence of the Hermite series, choosing appropriate values for the "free" parameters u_s, v_s, w_s and $\alpha_s, \beta_s, \gamma_s$. Another advantage is that one can in principle compute the full spectrum of eigenvalues (but for practical applications a subset of it is sufficient), with no chance of missing the most unstable mode. The method is therefore capable of computing both stable and unstable modes. Moreover, any physical quantity of interest (perturbed density, pressure tensor, heat flux, etc.) is related to a specific coefficient $C_{n,m,p}$ and thus can be easily examined.

III. FREE PARAMETERS

So far nothing has been said about the six free parameters u_s, v_s, w_s and $\alpha_s, \beta_s, \gamma_s$ that have to be specified in Eq. (11). It is clear that a "good" choice of these values strongly influence the convergence of the Hermite expansion. This is because these parameters represent the center and the width of the exponential functions used to weigh the Hermite polynomials, as well as the center and the width of the argument of the polynomials.

In order to improve the convergence of the scheme one should choose the six free parameters such that the solution f_1 (represented by a truncated series) has the lowest error with respect to the exact solution, for a given number of

polynomials used. It is important, however, to emphasize that any choice of the parameters would lead to an accurate solution, if a sufficient number of polynomials in the Hermite series is employed. A clever choice of the 6 free parameters then has the advantage of minimizing the number of polynomials needed for achieving a certain accuracy of the solution, that results in saving of computational effort. More details are deferred to Appendix A, while here it is sufficient to recall that the purpose of minimizing a quadratic error is achieved when the center velocities are chosen as:

$$v_s = \frac{\int_{-\infty}^{\infty} v_y f_s^2 e^{\xi_x^2} d\xi}{\int_{-\infty}^{\infty} f_s^2 e^{\xi_x^2} d\xi},$$
(27)

and similarly for u_s and w_s (see Ref. 20). A similar formula can also be derived for the velocities α_s , β_s , γ_s . It is reported in Appendix A [Eq. (A16)], but we will not use it in the present paper.

Even though the derivation of formula (27) is rigorous, its application to our method is not always straightforward. This is because the evaluation of the ratio of the numerical integrals can result in noise, especially when both the numerator and denominator tend to be null. Furthermore we need to point out that v_s defined by Eq. (27) is *a priori* not a constant, but a function of *x*. One can extend the theory so far derived allowing the free parameters to be functions of *x*. The derivation does not present any additional difficulty and is sketched for completeness in Appendix B. In the present paper we will however consider only constant values for u_s, v_s, w_s and $\alpha_s, \beta_s, \gamma_s$.

Different strategies for choosing the free parameters can be followed. At the beginning we always set the free parameters to be the drift and thermal velocities relative to the equilibrium function f_0 . We solve the eigenvalue problem (17) for different values of N, in order to observe the convergence of the solution as more Hermite polynomials are considered. In the cases in which the convergence is not satisfactory, some of the free parameters are adjusted according to the following strategy. The main idea is to compute Eq. (27) for v_s (and similarly for u_s and w_s) and choose the new constant velocities to be as close as possible to the computed ones (that are in general not constant), at least in the region of greater interest, that is in the center of the sheet (since the solution goes to zero at the boundary). In Sec. IV, we will show how this method can improve the convergence of the lower hybrid drift instability test problem.

IV. RESULTS

The method presented in the paper has been tested for the 1D equilibrium known as the Harris current-sheet described in Ref. 21. In this case, the unperturbed distribution function for each species is the drift-Maxwellian

$$f_{0s}(x, \boldsymbol{v}) = \frac{n(x)}{\pi^{3/2} v_{\text{th},s}^3} \exp\left[\frac{v_x^2 + (v_y - U_s)^2 + v_z^2}{v_{\text{th},s}^2}\right],$$
 (28)

where all quantities are dimensionless. The density n(x) is defined as

$$n(x) = \operatorname{sech}^2(x). \tag{29}$$

The distribution functions f_{0s} are the solutions of the Vlasov equation for an initial configuration with null electric field and with the following magnetic field profile:

$$B_{z0} = \tanh(x). \tag{30}$$

The following relation between the drift velocities holds: $U_i/T_i = -U_e/T_e$, so that the equilibrium is completely determined by choosing the four dimensionless parameters:

$$\frac{T_i}{T_a} = \frac{m_i}{m_a} = \frac{\rho_i}{L} = \frac{\omega_{pe}}{\Omega_{ca}}$$

where $\rho_i = v_{\text{th},i} / \Omega_{ci}$ is the ion gyroradius.

The last term in Eq. (12), which depends on the partial derivative of f_0 , becomes:

$$-\frac{\Omega_{cs}}{2^{q+r+t}q!r!t!}\int (E_{1}+v\times B_{1})\cdot\frac{\partial f_{0}}{\partial v}H_{q}(\xi_{s})H_{r}(\xi_{y})H_{t}(\xi_{z})d\xi$$

$$=\frac{\Omega_{cs}n(x)}{v_{\text{th},s}q!r!t!}\left\{qE_{x1}\left(-\frac{u_{s}}{v_{\text{th},s}}\right)^{q-1}\left(\frac{U-v_{s}}{v_{\text{th},s}}\right)^{r}\left(-\frac{w_{s}}{v_{\text{th},s}}\right)^{t}+rE_{y1}\left(-\frac{u_{s}}{v_{\text{th},s}}\right)^{q}\left(\frac{U-v_{s}}{v_{\text{th},s}}\right)^{r-1}\left(-\frac{w_{s}}{v_{\text{th},s}}\right)^{t}$$

$$+tE_{z1}\left(-\frac{u_{s}}{v_{\text{th},s}}\right)^{q}\left(\frac{U-v_{s}}{v_{\text{th},s}}\right)^{r}\left(-\frac{w_{s}}{v_{\text{th},s}}\right)^{t-1}-UtB_{x1}\left(-\frac{u_{s}}{v_{\text{th},s}}\right)^{q}\left(\frac{U-v_{s}}{v_{\text{th},s}}\right)^{t-1}$$

$$+UqB_{z1}\left(-\frac{u_{s}}{v_{\text{th},s}}\right)^{q}\left(\frac{U-v_{s}}{v_{\text{th},s}}\right)^{r}\left(-\frac{w_{s}}{v_{\text{th},s}}\right)^{t-1}\right\}.$$
(31)

The Harris equilibrium is unstable to a variety of instabilities including collisionless tearing, 22,23 drift tearing, 24,25 the lower-hybrid drift instability, $^{26-28}$ the drift kink

instability,⁸ and the ion-ion kink instability.^{8,29} In certain parameter regimes, some of these instabilities may be approximately modeled with fluid theory and researchers have in-



FIG. 1. Drift kink instability: Growth rate (top) and imaginary part of the eigenvalue (bottom) for different orders N in comparison to the result by Daughton (solid line).

ferred, based on numerical fluid calculations, that all of the current aligned drift modes may be closely related (see Ref. 30). However, fluid predictions are often unreliable in ion-scale current layers and a full kinetic treatment is usually required. Since the main focus of this work is to carefully benchmark the new method, we compare the results with a





FIG. 2. Drift kink instability: Relative error of the eigenvalue with respect to the previous order: $|\omega_N - \omega_{N-1}| / |\omega_N|$.

full linear Vlasov approach based on the method of characteristics (see Ref. 10). In order to do so, we have implemented our method to run on a single-processor machine. The computation of eigenvalues and eigenfunctions is done through the package ARPACK that allows us to compute only a portion of the full spectrum.

In this comparison, the discussion is limited to the driftkink instability (DKI), the lower-hybrid drift instability (LHDI), and collisionless tearing.

A. Drift kink instability

The following parameters have been chosen for this case:

$$\frac{m_i}{m_e} = 16, \quad \frac{\rho_i}{L} = 1, \frac{\omega_{pe}}{\Omega_{ce}} = 3, \quad \frac{T_i}{T_e} = 1,$$
 (32)

with wavelength $k_y=0.78$, $k_z=0$. Moreover we have adopted the following choice of the free parameters:

FIG. 3. Drift kink instability. Top panel: Re A_x (left) and Im A_x (right) for N=12, compared with the solution by Daughton. Bottom panel: Difference between subsequent orders $|A_{x,N} - A_{x,N-1}|$ for real part (left) and imaginary part (right).



FIG. 4. Drift kink instability: Top panel: Re A_y (left) and Im A_y (right) for N=12, compared to the solution by Daughton. Bottom panel: Difference between subsequent orders $|A_{y,N} - A_{y,N-1}|$ for real part (left) and imaginary part (right).

 $u_s = 0$, $v_s = U_s$, $w_s = 0$, $\alpha_s = \beta_s = \gamma_s = v_{\text{th},s}$.

Already using few orders of Hermite polynomials, we observed that all coefficients $C_{q,r,t}$ with $t \neq 0$ are null, meaning that this instability does not produce any kind of perturbation in the v_z space. Therefore it is sufficient to set t=0 for all the subsequent runs. This is a known property of any instability with $k_z=0$ [see Eqs. (16) and (17) in Ref. 8].

The code has run with 800 points in x and with orders N ranging from N=3 to N=14. Hereafter we denote with N the order of the coefficient $C_{q,r,t}$ defined as the sum of the three indexes. When N=14 the system (12) is composed of 120 equations per species, resulting in a matrix \hat{M} with a size of (198 152 × 198 152) and with 3 447 284 non-null elements.

Figure 1 shows the convergence of growth rate and of imaginary part, as a function of *N* compared with the result by Daughton: $\omega = 0.67 + 0.296i$. One can notice good agreement with the solution obtained by Daughton when N > 10. Figure 2 shows the relative error on the eigenvalue with respect to the previous order: $\varepsilon = |\omega_N - \omega_{N-1}| / |\omega_N|$. It shows that indeed the method is converging as *N* is increased, and when the order becomes higher than 10, the relative error is less than 0.5%. It is interesting to observe that the convergence seems to be dependent on the parity of the order considered. This is an argument that needs further investigation and is left as an open question here. It should be pointed out, however, that the parity of the order is not related to the parity of



FIG. 5. Drift kink instability: Top panel: Re ϕ (left) and Im ϕ (right) for N=12, compared with the solution by Daughton. Bottom panel: Difference between subsequent orders $|\phi_N - \phi_{N-1}|$ for real part (left) and imaginary part (right).



FIG. 6. (Color online) Drift kink instability: Perturbed distribution functions in the plane (v_x, v_y) at x=0,-0.1, -0.2,-0.3,-0.4,-0.5 for ions (top) and electrons (bottom). Both distribution functions are normalized with respect to the maximum value of $|f_i|$.

the solution. The order N is in fact defined as the maximum sum of the 3 indexes of the coefficients $C_{q,r,t}$ allowed to be nonvanishing. It follows that solutions of any order can correctly describe both even and odd physical quantities.

The eigenfunctions for the potentials A_x , A_y , ϕ , obtained with N=12, are shown, respectively, in the upper panel of Figs. 3–5, while the lower panel shows the difference between two eigenfunctions of subsequent orders for N=10, 11, 12. A_z is not shown, being null everywhere.

The results for A_x and A_y (upper part of Figs. 3 and 4) are hardly discernible from the solution by Daughton. A slight difference can however be noticed in the result for ϕ (Fig. 5), but the overall agreement is good. This is due to the

fact that the mode is nearly pure electromagnetic and therefore the convergence on ϕ is more problematic. In fact the maximum value of the scalar potential is roughly two orders of magnitude smaller than the maximum value of the vector potential A_x .

Figure 6 shows the absolute value of the perturbed distribution function for ions and electrons in the (v_x, v_y) plane for different values of x and with $v_z=0$ (N=12). The main feature, for both ions and electrons, is that far from the center of the sheet the distribution functions are nearly drifted-Maxwellian, while two beams tend to form when x approaches the center. For x=0 two distinctive anisotropic



FIG. 7. (Color online) Drift kink instability: Components of the pressure tensor for ions (left) and electrons (right). The top panel is component xx, the middle panel is component yy, and the lower panel is component xy. Pressure are normalized with respect to the maximum value of Re p_{xxe} .

beams are clearly visible. The distribution function is symmetric in the x direction and therefore only planes for x < 0 are shown.

and *xy* are shown in Fig. 7 (the other components being zero). These components have been obtained by multiplying by e^{ik_yy} , and considering only the real part. While the diagonal components present a quadrupolar structure, the off-

The real part of the pressure tensor, components xx, yy,



FIG. 8. Drift kink instability: Growth rate (top) and imaginary part of the eigenvalue (bottom), compared with results by Daughton. Mass ratio from $m_i/m_e=16$ to $m_i/m_e=400$.

diagonal component xy has a bipolar structure. However all the components, both for ions and for electrons, are displaced in the center of the sheet. Electron pressure is typically two orders of magnitude greater than the ion pressure.

Figure 7 has been obtained for N=12. We have also studied the dependence of ω as a function of the mass ratio. Figure 8 shows the well-known fact that the growth rate of the drift kink instability decreases as the mass ratio increases. For mass ratio between 16 and 400 the value of growth rate and imaginary part of the eigenvalue are in good agreement with the results obtained in Ref. 8. It should be pointed out that, for these results, ϕ was not fully converged (while A_x and A_y are) in light of the fact that the mode is more and more electromagnetic at higher mass ratios.

B. Lower hybrid drift instability

The lower hybrid drift instability is a perturbation with high frequency driven by the presence of a density gradient at the edge of the sheet. It has been extensively studied with relation to the problem of reconnection (see, for example, Refs. 26 and 31–33). As for the drift kink instability, the v_z



FIG. 9. Lower hybrid drift instability: Growth rate (top) and imaginary part of the eigenvalue (bottom) for different orders N in comparison with the result by Daughton. Circles are results for the first run. Crosses are results for the second run.

component of distribution function is not perturbed, and therefore we always set t=0.

In order to test the method in the presence of LHDI, the same parameters of the Harris current sheet used in Ref. 10 have been used:



FIG. 10. Lower hybrid drift instability: Relative error of eigenvalue with respect to the previous order $|\omega_N - \omega_{N-1}| / |\omega_N|$.



FIG. 11. Lower hybrid drift instability: Real part (top) and imaginary part (bottom) of v_i computed from Eq. (29) (solid), used in the first run (dashed), and adjusted for the second run (dotted).

$$\frac{\rho_i}{L} = 2, \quad \frac{m_i}{m_e} = 512, \quad \frac{\omega_{pe}}{\Omega_{ce}} = 5, \quad \frac{T_i}{T_e} = 1.$$
 (33)

The wavelengths are $k_y = 11.3$ and $k_z = 0$.

For this instability, two modes with very close growth rate, but with different parity, exist. These mode have been observed by Daughton and found also with our method here. Here, we present only the results for one of the two modes, corresponding to even parity for A_x and odd parity for A_y and ϕ . In this case the growth rate found by Daughton is $\omega_r/\Omega_{ci}=10.6$ and the imaginary part is $\omega_i/\Omega_{ci}=30.9$.

For this case we first run the code with 800 points in x and with the free parameters u_s, v_s, w_s and $\alpha_s, \beta_s, \gamma_s$ chosen to be equal to the drift and thermal velocities of the equilibrium function f_0 . N is ranging from N=3 to N=12. When N=12 the matrix \hat{M} has a size (151 810×151 810) and 2 575 486 non-null elements.

This first run gave a result that seemed to converge very slowly, as shown in Fig. 9, with circles.

If one looks at the relative error (Fig. 10) it is however less than 5% for N>9, because even if the growth rate has not yet converged, the imaginary part of the eigenvalue already did very well.

This first run was improved applying the technique explained in Sec. III and in Appendix A, by means of the following procedure.

The velocity v_i was computed by Eq. (27) using the solution with N=12, and its real and imaginary part are shown in Fig. 11. It is evident that the value $v_i=U_i$ considered for the first run, and plotted in Fig. 11 with the dashed line, was not the best choice and therefore could be adjusted. For the second run we considered v_i as the average between the values of the computed curve at x=0, and the value where the quadratic error ε_N (defined in Appendix A) is



FIG. 12. Lower hybrid drift instability. Eigenfunctions for different order N, compared to the solution by Daughton. Top panel: Re A_x (left) and Im A_x (right). Middle panel: Re A_y (left) and Im A_y (right). Bottom panel: Re ϕ (left) and Im ϕ (right).



FIG. 13. (Color online) Lower hybrid drift instability: components of the pressure tensor for ions (left) and electrons (right). The top panel is component xx, the middle panel is component yy, and the lower panel is component xy. Pressures are normalized with respect to the maximum value of Re p_{xxe} .

maximum ($x = \pm 1.3125$). v_i is considered as a complex number. The reader can be persuaded that the second run fully converged looking again at the eigenvalue in Fig. 9 (cross). Furthermore it clearly converges to the solution obtained by Daughton, and the relative error is smaller with respect to the first run (Fig. 10).

We report the eigenfunctions A_x , A_y , and ϕ compared with solution by Daughton, for N=12 in Fig. 12. Some noise is now observed in A_y near the center of the sheet, however the overall agreement is good. The well-known feature that the instability localizes at the edge of the sheet emerges clearly. This feature is also depicted in Fig. 13, where the



FIG. 14. (Color online) Lower hybrid drift instability: perturbed distribution functions in the plane (v_x, v_y) at x = 0, -0.1, -0.2, -0.3, -0.4, -0.5 for ions (top) and electrons (bottom). Both distribution functions are normalized wit respect to the maximum value of $|f_i|$.

components of the pressure tensor are shown. Again the electron pressure is dominant, the greater component being the diagonal term xx.

The perturbed distribution functions present much more complicated isosurfaces than in the drift kink case. They are again plotted in the (v_x, v_y) plane for several x and for $v_z=0$ in Fig. 14. While the ion distribution is composed by two beams at x=0, the electron one is now divided in four nearly isotropic beams. All the perturbation of the distribution function take place within x=0.5, remaining drifting Maxwellian further away.

C. Tearing instability

The tearing instability has been studied for the following parameters:

$$\frac{\rho_i}{L} = 2, \quad \frac{m_i}{m_e} = 16, \quad \frac{\omega_{pe}}{\Omega_{ce}} = 3, \quad \frac{T_i}{T_e} = 1$$
 (34)

in a configuration where the Harris current sheet is modified by the presence of a strong guide field $B_{y0}=5$. Wavelengths are

 $k_y=0, k_z=0.5$. Unlike the DKI and the LHDI, perturbations



FIG. 15. Tearing instability: Growth rate for different orders of N in comparison to the result by Daughton.

of the distribution functions are also along v_z . For this reason, the computation is now more expensive (the matrix \hat{M} is bigger) and therefore we only use 200 points in *x*. For this case, with N=9, the matrix \hat{M} has a size 89 152×89 152 and 1 511 532 non-null elements.

For the present case, the method devised by Daughton gives the result $\omega/\Omega_{ci}=0.255$ (with no imaginary part). Our result, for orders from N=3 to N=9 is plotted for comparison in Fig. 15. The relative error between subsequent orders is very small, and the solution converges to $\omega/\Omega_{ci}=0.256$. It is shown in Fig. 16, and it can be noticed that the error decreases under 0.1% already for N>5.

Eigenfunctions A_x , A_y , A_z , and ϕ are shown, compared with the result by Daughton in Figs. 17–20. Again the results agree very well.





FIG. 16. Tearing instability: Relative error of eigenvalue with respect to the previous order $|\omega_N - \omega_{N-1}|/|\omega_N|$.

Interesting features of the perturbed distribution functions in the center of the sheet are shown in Figs. 21-23 in which even with small mass ratio m_i/m_e , a different dynamics for electron and ions is present. Figure 21 shows slices of the absolute value of distribution functions in the (v_x, v_y) plane, for several values of v_z . The plot is very similar to that relative to the drift kink instability, in which a double beam distribution is present for $v_z=0$. This peculiarity is however much more pronounced for electrons than for ions. For higher values of v_{τ} both distributions are gyrotropic. Looking at what happens in the (v_x, v_z) plane (Fig. 22), electrons remain isotropic for any value of v_{y} , while ions present a double beam in correspondence of the value $v_y = -2$. Finally, a different dynamics for ions and electrons is also visible in the (v_y, v_z) plane (Fig. 23). Both species present a double beam when $v_x=0$, but while the beam is symmetric with



FIG. 17. Tearing instability: Top panel: Re A_x (left) and Im A_x (right) for different order *N*, compared to the solution by Daughton. Bottom panel: Difference between subsequent orders $|A_{x,N}-A_{x,N-1}|$ for the real part (left) and imaginary part (right).



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FIG. 18. Tearing instability: Re A_y (left) and Im A_y (right) for different order N, compared to the solution by Daughton. Bottom panel: Difference between subsequent orders $|A_{y,N} - A_{y,N-1}|$ for real part (left) and imaginary part (right).

respect to v_z for the ions, it is symmetric with respect to v_y for electrons. Moreover the latter seems not the be dependent on v_x .

Figure 24 shows the real part of the total density, imposed on the contour plot of the magnetic field in the (x,z) plane. It clearly shows a quadrupolar structure, and the typical signature of reconnection mediated by the kinetic Alfvén wave. Figure 25 shows the y and z component of magnetic field, while Fig. 26 plots the y component of the electric field.

V. CONCLUSION

A technique for solving the linearized Vlasov-Maxwell equations has been proposed. The technique is an alternative to the traditional method of integrating along the unperturbed orbits. It does not involve the knowledge of the particle orbits in the unperturbed fields, nor any integration along these orbits. In this sense it stands on much more easier physical and mathematical concepts, since the computation of these orbits is known to be very challenging.



FIG. 19. Tearing instability: $\operatorname{Re} A_z$ (left) and $\operatorname{Im} A_z$ (right) for different order *N*, compared with the solution by Daughton. Bottom panel: Difference between subsequent orders $|A_{z,N} - A_{z,N-1}|$ for real part (left) and imaginary part (right).



FIG. 20. Tearing instability: Top panel: Re ϕ (left) and Im ϕ (right) for different order *N*, compared with the solution by Daughton. Bottom panel: Difference between subsequent orders $|\phi_N - \phi_{N-1}|$ for real part (left) and imaginary part (right).

The method consists of the description of the velocity dependence of the perturbed distribution function by means of an infinite series of Hermite polynomials. The orthogonality properties of such a complete basis allow us to transform the Vlasov-Maxwell equations into an infinite set of linear coupled equations for the coefficients of the expansion. The problem is thus transformed in an eigenvalue problem.

The series is truncated with a closure relation. Nevertheless, an high order of accuracy has been proven in all the tests conducted.

The method has been tested in comparison to the traditional method devised by Daughton and described in Refs. 8–10 for the following instabilities: drift kink, lower hybrid drift, tearing. The results agreed well for all the three cases, showing the well-known features of these three instabilities. When the solutions seemed to converge slowly, the technique explained in Sec. III and in Appendix A was successfully applied in order to increase the convergence rate. An interesting feature of our method is that the accuracy of the solution seems dependent on the parity of the order considered, especially for the tearing instability. This result needs further investigation and will be left for future work.

The main computational difficulty of the method is related to the computation of the eigenvalues of a large sparse, complex matrix. We use the package ARPACK on a singleprocessor machine in order to compute a subset of the whole spectrum of eigenvalues. In this way, we have been able to reach an order N=14 with 800 points in x for the drift kink and the lower hybrid drift instabilities, and N=9 with 200 points in x for the tearing instability. For the LHDI case, an adjustment of some of the six free parameters was needed, in order to achieve a better convergence (that is, convergence with fewer Hermite polynomials) and to overcome computational limits. The general idea to follow for the adjustment of the free parameters has been elucidated. A future perspective of the present work could be to include a collisional term in the linearized Vlasov equation; a feature that the traditional method of characteristics is unable to solve.

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APPENDIX A: CHOICE OF FREE PARAMETERS

In this section, we show how to determine the "best" choice for the functions u, v, w, and α , $\beta \gamma$ of Eq. (12) in order to obtain the lowest error in the truncation of the series (we have dropped the subscript *s* to refer to the species). The derivation follows the theory devised in Ref. 34 (see also Ref. 20). For the sake of clarity, we here consider a function

$$f(v) = \sum_{n=0}^{\infty} C_n H_n \left(\frac{v-u}{v_{\rm th}}\right) e^{-(v-u/v_{\rm th})^2},$$
 (A1)

where H_n is the Hermite polynomial of order *n*, but the derivation can be easily extended to the case of Eq. (12).



FIG. 21. (Color online) Tearing instability: perturbed distribution functions in the plane (v_x, v_y) at $v_z = -3$, -1.5, 0, 1.5, 3 for ions (top) and electrons (bottom).

We define an auxiliary function ϕ_n :

 $\phi_n(\xi) = H_n(\xi)e^{-\xi^2} \tag{A2}$

with $\xi = (v - u)/v_{\text{th}}$, and an inner product

$$\langle \phi_n, \phi_m \rangle = \int_{-\infty}^{\infty} \omega(\xi) \phi_n \phi_m d\xi.$$
 (A3)

The weighting function is chosen as $\omega(\xi) = e^{\xi^2}$ in order to achieve the usual orthogonality relation

 $\langle \phi_n, \phi_m \rangle = \delta_{n,m} 2^n n!$ (δ is the Kronecker symbol).

The truncated series is defined as

$$f_N(v) = \sum_{n=0}^{N-1} C_n \phi_n(\xi).$$
 (A5)

We want to minimize with respect to u the quadratic error ε_N defined in the following way:



FIG. 22. (Color online) Tearing instability: Perturbed distribution functions in the plane (v_x, v_z) at $v_y=1,-0.5$, -2,-3.5,-5 for ions (top) and v_y = 4,2.5,1,-0.5,-2 electrons (bottom).

$$\varepsilon_N = \langle f - f_N, f - f_N \rangle = \int \omega(v) (f(v) - f_N(v))^2 dv$$
$$= \sum_{n=N}^{\infty} 2^n n! C_n^2. \tag{A6}$$

One can easily derive the first and second derivative of ϕ with respect to v:

$$\frac{\partial \phi_n}{\partial v} = \frac{e^{-\xi^2}}{v_{\rm th}} (H'_n - 2\xi H_n) \tag{A7}$$

$$\frac{\partial^2 \phi_n}{\partial v^2} = \frac{e^{-\xi^2}}{v_{\rm th}^2} (H_n'' - 4\xi H_n' + H_n(4\xi^2 - 2)), \tag{A8}$$

where a prime denotes derivative with respect to ξ .



FIG. 23. (Color online) Tearing instability: Perturbed distribution functions in the plane (v_y, v_z) at $v_x = -3$, -1.5, 0, 1.5, 3 for ions (top) and electrons (bottom).

Making now use of the relations of the Hermite polynomials as given in Ref. 35

$$H_n'' - 2\xi H_n' + 2nH_n = 0 (A9)$$

$$-v_{\rm th}^2 \frac{\partial^2 \phi_n}{\partial v^2} + 4\xi^2 \phi_n - H_n'' e^{-\xi^2} = (4n+2)\phi_n. \tag{A10}$$

If we now multiply both sides of Eq. (A10) by C_n , sum over n and calculate the inner product with f, for the right hand side of the previous equation we have:

one has



FIG. 24. (Color online) Tearing instability. Real part of the total density and contour plot of the magnetic field.

$$\varepsilon_N^{\max} = -v_{th}^2 m_3 + \frac{4}{v_{th}^2} (m_2 - 2m_1 u + m_0 u^2) - m_4.$$
(A13)

Finally, minimizing the previous relation we find that the optimum value for u is

$$u = \frac{m_1}{m_0} = \frac{\int v f^2 e^{\xi^2} d\xi}{\int f^2 e^{\xi^2} d\xi}.$$
 (A14)

The same can be done with respect to the scale value $v_{\rm th}$, and in this case the error bound is minimized when

$$v_{\rm th}^4 = \frac{4}{m_3} (2m_1 u - m_0 u^2 - m_2) \tag{A15}$$

that becomes

$$v_{\rm th}^4 = 4 \frac{m_1^2 - m_2 m_0}{m_3 m_0} \tag{A16}$$

by using Eq. (A14). Note that these results are equal to those obtained in Ref. 20 using a slightly different form for f(v).

$$\left\langle f, \sum_{n=0}^{\infty} (4n+2)C_n \phi_n \right\rangle = \sum_{n=0}^{\infty} (4n+2) \cdot 2^n n! C_n^2$$
$$> \sum_{n=N}^{\infty} 2^n n! C_n^2 = \varepsilon_N.$$
(A11)

We therefore have an upper bound $\varepsilon_N^{\text{max}}$ for the error ε_N :

$$\varepsilon_N^{\max} = -v_{\rm th}^2 \int f \frac{\partial^2 f}{\partial v^2} e^{\xi^2} d\xi + \int 4\xi^2 f^2 e^{\xi^2} d\xi$$
$$-\int \left(\sum_{n=0}^{\infty} C_n H_n\right) \left(\sum_{q=0}^{\infty} C_q H_q''\right) e^{-\xi^2} d\xi. \tag{A12}$$

If we now define the following quantities:

$$\begin{split} m_{i} &= \int_{-\infty}^{\infty} v^{i} f^{2} e^{\xi^{2}} d\xi \quad i = 0, 1, 2, \\ m_{3} &= \int_{-\infty}^{\infty} f \frac{\partial^{2} f}{\partial v^{2}} e^{\xi^{2}} d\xi, \\ m_{4} &= \int \left(\sum_{n=0}^{\infty} C_{n} H_{n}\right) \left(\sum_{q=0}^{\infty} C_{q} H_{q}''\right) e^{-\xi^{2}} = \int \left(\sum_{n=0}^{\infty} C_{n} H_{n}\right) \\ &\times \left(\sum_{q=0}^{\infty} 4q(q-1)C_{q} H_{q-2}\right) e^{-\xi^{2}} \\ &= \int \left(\sum_{n=0}^{n=\infty} C_{n} H_{n}\right) \left(\sum_{q=2}^{\infty} 4(q+1)(q+2)C_{q+2} H_{q}\right) e^{-\xi^{2}} \\ &= \sum_{n=2}^{\infty} 4(n+1)(n+2)C_{n}C_{n+2}2^{n}n! \end{split}$$

the expression for $\varepsilon_N^{\text{max}}$ reads



FIG. 25. (Color online) Tearing instability. Real part of the out-of-plane field B_y (top) and real part of B_z (bottom). Contour plot of magnetic field is plotted in both figures.



FIG. 26. (Color online) Tearing instability. Real part of E_y and contour plot of the magnetic field.

Moreover, notice that the parameter u can be a function of the spatial coordinate x, as in our original formulation [Eq. (11)] without any change in Eq. (A14).

Generalizing now the previous derivation to our distribution function f_1 , we obtain the following values:

$$m_0 = \sum_{q,r,t} (C_{q,r,t})^2 2^{q+r+t} q! r! t!, \qquad (A17)$$

$$m_{1x} = \alpha \sum_{q,r,t} C_{q,r,t} \left(\frac{1}{2} C_{q-1,r,t} + (q+1) C_{q+1,r,t} \right) 2^{q+r+t} q! r! t!$$

+ um_0 , (A18)

$$\begin{split} m_{2x} &= \alpha^2 \sum_{q,r,t} C_{q,r,t} \bigg(\frac{1}{4} C_{q-2,r,t} + q C_{q,r,t} + (q+1) \\ &\times (q+2) C_{q+2,r,t} \bigg) 2^{q+r+t} q! r! t! - u(m_{1x} - um_0) \\ &+ \bigg(\frac{\alpha^2}{2} + u^2 \bigg) m_0, \end{split}$$
(A19)

$$m_{3x} = \frac{1}{\alpha^2} \sum_{q,r,t} C_{q,r,t} (C_{q-2,r,t} + 2(q-1)C_{q,r,t} + 4(q+1)^2 C_{q+2,r,t}) 2^{q+r+t} q! r! t!,$$
(A20)

and similarly for y and z components.

APPENDIX B: GENERALIZATION OF THE METHOD

The method devised in the present paper can be generalized taking in account the spatial dependence of the parameters u_s , v_s , w_s , and α_s , β_s , γ_s . In this case, Eq. (12) becomes (index s is dropped)

$$\begin{split} \frac{\partial C_{q,r,t}}{\partial t} &= -\alpha_s \bigg[\frac{\partial C_{q+1,r,t}}{\partial x} (q+1) + \frac{1}{2} \frac{\partial C_{q-1,r,t}}{\partial x} + \frac{u}{\alpha} \frac{\partial C_{q,r,t}}{\partial x} \bigg] - \beta i k_y \bigg[(r+1) C_{q,r+1,t} + \frac{1}{2} C_{q,r-1,t} + \frac{v}{\beta} C_{q,r,t} \bigg] \\ &\quad - \gamma i k_z \bigg[(t+1) C_{q,r,t+1} + \frac{1}{2} C_{q,r,t-1} + \frac{w}{\gamma} C_{q,r,t} \bigg] - \alpha' \bigg(\frac{1}{4} C_{q-3,r,t} + \bigg(q + \frac{1}{2} \bigg) C_{q-1,r,t} + (q+1)^2 C_{q+1,r,t} \bigg) - \bigg(u' + \frac{u\alpha'}{\alpha} \bigg) \\ &\quad \times \bigg(\frac{1}{2} C_{q-2,r,t} + (q+1) C_{q,r,t} \bigg) - \frac{uu'}{\alpha} C_{q-1,r,t} - \frac{\alpha v'}{\beta} \bigg((q+1) C_{q+1,r-1,t} + \frac{1}{2} C_{q-1,r-1,t} + \frac{u}{\alpha} C_{q,r-1,t} \bigg) \\ &\quad - \frac{\alpha \beta'}{\beta} \bigg[\frac{1}{4} C_{q-1,r-2,t} + \frac{(r+1)}{2} C_{q-1,r-1,t} + \frac{(q+1)}{2} C_{q+1,r-2,t} + (q+1)(r+1) C_{q+1,r-1,t} \bigg] \\ &\quad - \frac{u\beta'}{\beta} \bigg[\frac{1}{2} C_{q,r-2,t} + (r+1) C_{q,r-1,t} \bigg] - \frac{\alpha w'}{\gamma} \bigg((q+1) C_{q+1,r,t-1} + \frac{1}{2} C_{q-1,r,t-1} + \frac{u}{\alpha} C_{q,r,t-1} \bigg) \\ &\quad - \frac{\alpha \beta'}{\gamma} \bigg[\frac{1}{4} C_{q-1,r,t-2} + \frac{(r+1)}{2} C_{q-1,r,t-1} + \frac{(q+1)}{2} C_{q+1,r,t-2} + (q+1)(t+1) C_{q+1,r,t-1} \bigg] \\ &\quad - \frac{u\beta'}{\gamma} \bigg[\frac{1}{2} C_{q,r,t-2} + (r+1) C_{q,r,t-1} \bigg] - \frac{\alpha w'}{\gamma} \bigg((q+1) C_{q+1,r,t-2} + (q+1)(t+1) C_{q+1,r,t-1} \bigg] \\ &\quad - \frac{u_{\gamma}\gamma}{\gamma} \bigg[\frac{1}{2} C_{q,r,t-2} + (t+1) C_{q,r,t-1} \bigg] + \frac{\Omega_c}{\alpha} E_{x0} C_{q-1,r,t} + \frac{\Omega_c}{\beta} E_{y0} C_{q,r-1,t} + \frac{\Omega_c}{\gamma} E_{z0} C_{q,r,t-1} \\ &\quad - \Omega_c B_{x0} \bigg[\frac{\beta}{\gamma} C_{q,r+1,t-1}(r+1) - \frac{\gamma}{\beta} C_{q,r-1,t+1}(t+1) + \bigg(\frac{\beta}{\gamma} - \frac{\gamma}{\beta} \bigg) \frac{1}{2} C_{q-1,r,t-1} + \frac{w}{\gamma} C_{q,r,t-1} - \frac{w}{\gamma} C_{q,r,t-1} \bigg] \\ &\quad - \Omega_c B_{y0} \bigg[\frac{\gamma}{\alpha} C_{q-1,r,t+1}(t+1) - \frac{\alpha}{\gamma} C_{q+1,r,t-1}(q+1) + \bigg(\frac{\alpha}{\gamma} - \frac{\alpha}{\gamma} \bigg) \frac{1}{2} C_{q-1,r,t-1} + \frac{w}{\beta} C_{q,r-1,r} - \frac{w}{\gamma} C_{q,r,t-1} \bigg] \\ &\quad - \Omega_c B_{y0} \bigg[\frac{\alpha}{\beta} C_{q+1,r-1,t}(q+1) - \frac{\beta}{\alpha} C_{q-1,r,t-1}(q+1) + \bigg(\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \bigg) \frac{1}{2} C_{q-1,r,t-1} + \frac{w}{\beta} C_{q,r-1,r} \bigg] \bigg]$$

$$-\frac{\omega_c}{2^{q+r+t}q!r!t!}\int (\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_1) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} H_q(\xi_x) H_r(\xi_y) H_t(\xi_z) d\xi, \tag{B1}$$

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where prime denotes the derivation with respect to x. For the particular case of the Harris sheet used as benchmark in the present paper, the last term (that depends on the partial derivative of f_0) can be computed by means of the following formulas (see Ref. 35), with the usual convention $0^0 = 1$:

$$\int e^{-(x-y)^2} H_n(\alpha x) dx = \sqrt{\pi} (1-\alpha^2)^{n/2} H_n\left[\frac{\alpha y}{(1-\alpha^2)^{1/2}}\right],$$
(B2)

$$e^{-(x-y)^{2}}H_{m}(\alpha x)H_{n}(\alpha x)dx$$

$$=\sqrt{\pi}\sum_{k=0}^{\min(m,n)}2^{k}k!\binom{m}{k}\binom{n}{k}$$

$$\times(1-\alpha^{2})^{m+n/2-k}H_{m+n-2k}\left[\frac{\alpha y}{(1-\alpha^{2})^{1/2}}\right],$$
(B3)

$$H_n(x) = 2^n x^n - 2^{n-1} \binom{n}{2} x^{n-2} + 2^{n-2} \cdot 1 \cdot 3 \binom{n}{4} x^{n-4} - 2^{n-3} \cdot 1 \cdot 3 \cdot 5 \binom{n}{6} x^{n-6} + \dots$$
(B4)

that is

$$e^{-(x-y)^{2}}H_{n}(\alpha x)dx$$

= $\sqrt{\pi}\sum_{i=0,2,4,\dots}^{q} \frac{2^{q-i/2}q!(i+1)!!}{(i+1)!(q-i)!}(\alpha y)^{q-i}(1-\alpha^{2})^{i/2},$ (B5)

$$\int e^{-(x-y)^2} H_1(\alpha x) H_q(\alpha x) dx = \sqrt{\pi} \left[(1-\alpha^2)^{q+1/2} H_{q+1} \left[\frac{\alpha y}{(1-\alpha^2)^{1/2}} \right] + 2q(1-\alpha^2)^{q-1/2} H_{q-1} \left[\frac{\alpha y}{(1-\alpha^2)^{1/2}} \right] \right]$$
$$= \sqrt{\pi} \left[\sum_{i=0,2,4,\dots}^{q+1} \frac{2^{q+1-i/2}(q+1)!(i+1)!!}{(i+1)!(q+1-i)!} (\alpha y)^{q+1-i} (1-\alpha^2)^{i/2} + 2q \sum_{i=0,2,4,\dots}^{q-1} \frac{2^{q-1-i/2}(q-1)!(i+1)!!}{(i+1)!(q-1-i)!} (\alpha y)^{q-1-i} (1-\alpha^2)^{i/2} \right].$$
(B6)

Finally, the last term of Eq. (B1) becomes

$$-\frac{\Omega_{c}}{2^{q+r+t}q!r!t!} \int (E_{1} + \mathbf{v} \times \mathbf{B}_{1}) \cdot \frac{\partial f_{0}}{\partial \mathbf{v}} H_{q}(\xi_{x}) H_{r}(\xi_{y}) H_{t}(\xi_{z}) d\xi$$

$$= +\frac{\Omega_{cs}n(x)}{2^{q+r+t}q!r!t!} \left\{ \frac{v_{th}}{\gamma\beta} E_{x1} \left(S_{x}^{q+1} + 2qS_{x}^{q-1} + \frac{2u}{\alpha} S_{x}^{q} \right) S_{y}^{r} S_{z}^{t} + \frac{v_{th}}{\alpha\gamma} E_{y1} \left(S_{y}^{r+1} + 2rS_{y}^{r-1} + \frac{2(v-U)}{\beta} S_{y}^{r} \right) S_{x}^{q} S_{z}^{t}$$

$$+ \frac{v_{th}}{\alpha\beta} E_{z1} \left(S_{z}^{t+1} + 2tS_{z}^{t-1} + \frac{2w}{\gamma} S_{z}^{t} \right) S_{x}^{q} S_{y}^{r} - \frac{v_{th} UB_{x1}}{\alpha\beta} \left[\left(S_{z}^{t+1} + 2tS_{z}^{t-1} + \frac{2w}{\gamma} S_{z}^{t} \right) S_{x}^{q} S_{y}^{r} \right]$$

$$+ \frac{v_{th} UB_{z1}}{\gamma\beta} \left[\left(S_{x}^{q+1} + 2qS_{x}^{q-1} + \frac{2v}{\alpha} S_{x}^{q} \right) S_{z}^{t} S_{y}^{r} \right] \right\}, \qquad (B7)$$

where the following quantities are defined:

$$S_{z}^{t} = \sum_{k=0,2,4,\dots}^{t} \frac{2^{t-k/2}t!(k+1)!!}{(k+1)!(t-k)!} (-w/\gamma)^{t-k} \left(1 - \left(\frac{v_{\text{th}}}{\gamma}\right)^{2}\right)^{k/2}$$

and k!! = k(k-2)(k-4)...

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