IOP Publishing Plasma Physics and Controlled Fusion

Plasma Phys. Control. Fusion 61 (2019) 054005 (7pp) https://doi.org/10.1088/1361-6587/ab04d5

Fourier–Hermite decomposition of the collisional Vlasov–Maxwell system: implications for the velocity-space cascade

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Received 2 November 2018, revised 16 January 2019
Accepted for publication 6 February 2019
Published 21 March 2019

Abstract

Turbulence at kinetic scales is an unresolved and ubiquitous phenomenon that characterizes both space and laboratory plasmas. Recently, new theories, *in situ* spacecraft observations and numerical simulations suggest a novel scenario for turbulence, characterized by a so-called phase-space cascade—the formation of fine structures, both in physical and velocity-space. This new concept is here extended by directly taking into account the role of inter-particle collisions, modeled through the nonlinear Landau operator or the simplified Dougherty operator. The characteristic times, associated with inter-particle correlations, are derived in the above cases. The implications of introducing collisions on the phase-space cascade are finally discussed.

Keywords: plasma turbulence, kinetic plasmas, space plasmas, collisions

1. Introduction

Hot and dilute plasmas, which are ubiquitous in near-Earth environments and in astrophysical systems, often exhibit a strongly turbulent and complex dynamics [1–3]. The fluctuations energy, injected at large scales, is typically transferred to smaller scales, enabling the energy transfer to particles. This process ultimately ceases by dissipating the energy and heating the plasma [4]. When the energy of the fluctuations is available to be transferred to the distribution function of particles, it is possible to recover dynamical states which are very far from the thermal equilibrium. As a consequence, the particles distribution function is strongly deformed and exhibits temperature anisotropies, rings and beam-like structures as well as velocity-space jets and vortices [5–11]. Even though these systems are usually described by means of collisionless models, the presence of fine structures (i.e. strong gradients) in velocity-space may enhance the role of collisions [12, 13]. Collisions may therefore result as one of the possible ingredients contributing to plasma heating [12, 14, 15]. It should be highlighted that collisions have the intrinsic characteristic of making the system irreversible. Hence, it is possible—when considering collisional effects—to heat the system in a pure thermodynamic sense, intimately related to the irreversible degradation of the information.

Extensive efforts have been devoted to understand whether and how the turbulent cascade, routinely depicted in physical space, evolves also in velocity-space. To better appreciate the presence of phase-space fluctuations, the particle distribution function is often decomposed in terms of Hermite polynomials for the velocity-space variables, while the usual Fourier decomposition is adopted to describe fluctuations in physical space. Since the seminal work of Grad [16], numerous results have invoked the Hermite decomposition of the velocity distribution function. Several analyses have been focused on the description of the Vlasov–Poisson system in the collisionless case [17, 18] or by modeling collisions with the Lenard–Bernstein operator [19–26]. This system has been also studied extensively from a numerical viewpoint [27–33]. Other important studies have been dedicated to the description of phase-space fluctuations in the framework of gyrokinetics and drift-wave turbulence [34–41] as well as, for fusion research, ion temperature gradient (ITG) driven turbulence [42, 43]. Even for this latter category, collisions are usually modeled through simplified collisional operators, such as the Lenard–Bernstein model. Very recently, an extensive characterization of the phase-space cascade has been proposed by Eyink [44].

Separate studies have instead focused on modeling collisional effects by means of collisional integrals, such as the Landau one [45], which can be derived from 'first-principle' arguments. In particular, the Hermite moments of the Landau integral and its gyrokinetics formulation have been recently discussed [46–48]. The analysis has been also extended to the hybrid or full Vlasov–Maxwell system. This has been done for numerical objectives [49] and also for describing space plasmas [50–52]. Indeed, complete models are crucial to describe kinetic turbulence in space plasmas, where complex phenomena—such as, for example, magnetic reconnection, stationary (zero-frequency) current structures and nonlinear damping—emerge at different spatial and temporal scales and interact [53–67].

The Hermite decomposition of the particle distribution function has been adopted to characterize the velocity-space cascade. This picture indeed resembles a process of cascade where free energy is injected at large scales (low Hermite coefficients) and is transferred to higher Hermite modes, analogously to the Fourier counterpart for the physical space cascade of fluids. At small velocity scales, finally, collisions ultimately provide the channel for dissipating these fluctuations. For the first time, in Servidio *et al* [50], thanks to the high-resolution measurements of magnetospheric multiscale mission [68], the velocity-space cascade has been directly observed in the Earth's magnetosheath. A novel collisionless theory, based on the Kolmogorov approach, has been also developed and theoretical predictions are in accordance with both *in situ* observations [50] and numerical results obtained within the hybrid Vlasov–Maxwell framework [51, 52].

Here we extend the theory proposed in [50] by taking into account the role of inter-particle collisions. Collisions are modeled through the Landau operator [45, 69], which can be derived from the Liouville equation, or, alternatively, the Dougherty operator [70–72], which is an 'ad hoc' simpler operator, that is still nonlinear and obeys the *H*-theorem. The Dougherty operator has been recently compared to the Landau one [73] and adopted for performing self-consistent Eulerian simulations [74, 75]. The two operators are here written in the Fourier–Hermite space and the differences between them are discussed in detail. It is shown that—in the asymptotic regime—the Landau operator shows two characteristic times, respectively associated with the diffusive and the drag part of the operator. The fastest characteristic time related to the Landau operator scales as m^{-2}, being m the Hermite coefficient. On the other hand, the collisional characteristic time, when collisions are modeled through the Dougherty operator, is proportional to m^{-1}. For the Landau operator case, we derive the typical Hermite coefficients m^*, corresponding to the balance of the collisional characteristic time and the collisionless ones. We apply these results to typical natural and laboratory plasmas, finding that m^* is generally large. This implies that the phenomenological theory described by Servidio *et al* [50] holds its validity up to large Hermite coefficients.

As far as we know, the Fourier–Hermite decomposition of the Landau operator represents a novel result: previous works have been focused on the Hermite moments of the Landau operator [46–48], while here the relevant part of the discussion concerns the implication of including collisions in the phase-space cascade. Moreover, here we present a compact notation of the collisional Vlasov–Maxwell system in terms of annihilation and creation operators, widely adopted in the quantum mechanics framework.

The paper is organized as follows. In section 2, we revisit the Fourier–Hermite decomposition of the Vlasov–Maxwell system of equations, by ignoring collisions. Then, in section 3, we include the effect of collisions, modeled through both the Landau operator or, alternatively, the Dougherty operator. In section 4 we discuss the role of collisions in terms of velocity-space cascade by deriving the collisional characteristic time and by comparing it to the characteristic times associated with the collisionless part of the Vlasov equation. Finally, in section 5, a summary of the presented results is given.

2. Fourier–Hermite decomposition of Vlasov–Maxwell equations

The dynamics of a non-relativistic, quasi-neutral plasma in absence of inter-particle collisions can be described by means of the Vlasov–Maxwell system of equations, which in CGS units are:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \nabla_v f_\alpha = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{4}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}, \tag{5}$$

where $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ is the distribution function of the α-species, $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are respectively the electric and magnetic fields, q_α and m_α are the charge and mass of the α-species and $\rho(\boldsymbol{r}, t) = \sum_\alpha q_\alpha n_\alpha(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t) = \sum_\alpha q_\alpha n_\alpha(\boldsymbol{r}, t)\boldsymbol{u}_\alpha(\boldsymbol{r}, t)$ are the charge and current density, respectively. In the above definitions, $n_\alpha(\boldsymbol{r}, t) = \int d^3v f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$ and $\boldsymbol{u}_\alpha(\boldsymbol{r}, t) = \int d^3v\, \boldsymbol{v} f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t) / n_\alpha(\boldsymbol{r}, t)$ are respectively the α-species number density and bulk speed. For the sake of simplicity, the dependencies on $\boldsymbol{r}$, $\boldsymbol{v}$ and t of the several variables in equations (1)–(5) are omitted. Note that, at the right hand side of equation (1), no collisional operators are introduced.

In order to decompose equations (1)–(5), we adopt the asymmetrically weighted Hermite functions, whose three-dimensional orthonormal basis is:

$$\begin{cases} \boldsymbol{\Psi}_{\boldsymbol{m}}(\boldsymbol{\zeta}_\alpha) = \psi_{m_x}(\zeta_{\alpha,x})\psi_{m_y}(\zeta_{\alpha,y})\psi_{m_z}(\zeta_{\alpha,z}) \\ \boldsymbol{\Psi}^{\boldsymbol{m}}(\boldsymbol{\zeta}_\alpha) = \psi^{m_x}(\zeta_{\alpha,x})\psi^{m_y}(\zeta_{\alpha,y})\psi^{m_z}(\zeta_{\alpha,z}) \end{cases}, \tag{6}$$

where $\boldsymbol{\zeta}_\alpha = (\boldsymbol{v} - \boldsymbol{u}_{\alpha,0})/\sqrt{2}\,v_{\mathrm{th}\alpha,0}$, while $\boldsymbol{u}_{\alpha,0}$ and $v_{\mathrm{th}\alpha,0} = \sqrt{k_B T_{\alpha,0}/m_\alpha}$ are the α-species bulk and thermal speed at $t = 0$. We also assume that (a) each species is initially at rest ($\boldsymbol{u}_{\alpha,0} = 0$) and (b) the initial equilibrium is homogeneous ($T_{\alpha,0}$ and $n_{\alpha,0}$ are constant). Clearly, the variable $\boldsymbol{\zeta}_\alpha$ is independent from $\boldsymbol{r}$ and t.

In equations (6), ψ_m and ψ^m are respectively the covariant and controvariant one-dimensional Hermite functions:

$$\begin{cases} \psi_m(\zeta) = \frac{H_m(\zeta)\mathrm{e}^{-\zeta^2}}{\sqrt{2^m m!\pi}} \\ \psi^m(\zeta) = \frac{H_m(\zeta)}{\sqrt{2^m m!}} \end{cases} \tag{7}$$

being $H_m(\zeta) = (-1)^m \mathrm{e}^{\zeta^2}\mathrm{d}^m/\mathrm{d}\zeta^m \mathrm{e}^{-\zeta^2}$ the mth order 'physicists' Hermite polynomial. The following properties are satisfied for the Hermite functions defined in equations (7):

$$\begin{cases} \int \mathrm{d}\zeta \psi^m(\zeta)\psi_n(\zeta) = \delta_{m,n} \\ \zeta\psi_m(\zeta) = \sqrt{\frac{m+1}{2}}\psi_{m+1}(\zeta) + \sqrt{\frac{m}{2}}\psi_{m-1}(\zeta). \\ \frac{\mathrm{d}\psi_m(\zeta)}{\mathrm{d}\zeta} = -\sqrt{2(m+1)}\,\psi_{m+1}(\zeta) \end{cases} \tag{8}$$

The above defined basis is exploited to decompose the variables in equations (1)–(5) as follows:

$$\begin{cases} f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t) = \sum_{\boldsymbol{m},\boldsymbol{k}} \tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{\Psi}_{\boldsymbol{m}}(\boldsymbol{\zeta}_\alpha) \\ g(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} \tilde{g}_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \end{cases} \tag{9}$$

being g a generic function which depends only on $\boldsymbol{r}$ and t, such as ρ or any component of $\boldsymbol{E}$, $\boldsymbol{B}$ or $\boldsymbol{j}$.

To obtain the evolution equation for the Fourier–Hermite coefficient $\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}$, equation (1) is multiplied for $\boldsymbol{\Psi}^{\boldsymbol{m}}(\boldsymbol{\zeta}_\alpha)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3 r \mathrm{d}^3\zeta_\alpha$ and, then, the integral on the whole phase-space is evaluated. After some algebra, it is easy to obtain:

$$\frac{\partial \tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}}{\partial t} + \mathrm{i}v_{\mathrm{th}\alpha,0}\,\boldsymbol{k}\cdot(\hat{\boldsymbol{a}}^\dagger + \hat{\boldsymbol{a}})\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}} - \frac{q_\alpha}{m_\alpha v_{\mathrm{th}\alpha,0}} \times \sum_{\boldsymbol{k}_2}\left(\tilde{\boldsymbol{E}}_{\boldsymbol{k}-\boldsymbol{k}_2} + \frac{v_{\mathrm{th}\alpha,0}}{c}\hat{\boldsymbol{a}}^\dagger \times \tilde{\boldsymbol{B}}_{\boldsymbol{k}-\boldsymbol{k}_2}\right)\cdot\hat{\boldsymbol{a}}\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}_2} = 0. \tag{10}$$

Since in equation (1) the Lorentz-force term is nonlinear in space, the convolution over the Fourier wavevector $\boldsymbol{k}_2$ is recovered in equation (10). It is worth to highlight that, with respect to the notation adopted in previous papers [28, 49], here—in order to achieve a compact notation—we make use of the vectorial creation $\hat{\boldsymbol{a}}^\dagger$ and annihilation $\hat{\boldsymbol{a}}$ operators, defined as usual as:

$$\begin{cases} \hat{a}_j\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}} = \sqrt{m_j}\,\tilde{f}_{\alpha,\boldsymbol{m}-\boldsymbol{e}_j,\boldsymbol{k}} \\ \hat{a}_j^\dagger\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}} = \sqrt{m_j+1}\,\tilde{f}_{\alpha,\boldsymbol{m}+\boldsymbol{e}_j,\boldsymbol{k}} \end{cases} \tag{11}$$

being $j = x, y, z$ and $\boldsymbol{e}_i$ the ith Cartesian unit vector. Each time a creation $\hat{\boldsymbol{a}}^\dagger$ or annihilation $\hat{\boldsymbol{a}}$ operator is introduced, a factor proportional to $\sqrt{m}$ is recovered. Note also that in equation (10) only neighbor couplings of different Hermite modes are recovered.

3. Collisional operators in the Fourier–Hermite space

In the current section we extend the result of equation (10) by considering inter-particle collisions and, hence, evaluating

$$\tilde{C}_{\alpha,\boldsymbol{m},\boldsymbol{k}}(t) = \int \mathrm{d}^3\zeta_\alpha \mathrm{d}^3 r\,\boldsymbol{\Psi}^{\boldsymbol{m}}(\boldsymbol{\zeta}_\alpha)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}C_\alpha, \tag{12}$$

where C_α is the collisional operator.

3.1. Collisions modeled with the Landau operator

When focusing on the Landau operator, it is convenient to use the Landau operator written in terms of the Rosenbluth–MacDonald–Judd (RMJ) potentials [69]:

$$C_\alpha^{\mathrm{LAN}} = \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2 \ln\Lambda}{m_\alpha}\frac{\partial}{\partial v_i}\times \times\left[\frac{1}{2m_\alpha}\frac{\partial}{\partial v_j}\left(\frac{\partial^2 g_\beta}{\partial v_i \partial v_j}f_\alpha\right) - \frac{1}{\mu_{\alpha\beta}}\frac{\partial h_\beta}{\partial v_i}f_\alpha\right], \tag{13}$$

where

$$\begin{cases} g_\beta(\boldsymbol{r}, \boldsymbol{v}, t) = \int \mathrm{d}^3 v' f_\beta(\boldsymbol{r}, \boldsymbol{v}', t)|\boldsymbol{v} - \boldsymbol{v}'| \\ h_\beta(\boldsymbol{r}, \boldsymbol{v}, t) = \int \mathrm{d}^3 v' \frac{f_\beta(\boldsymbol{r}, \boldsymbol{v}', t)}{|\boldsymbol{v} - \boldsymbol{v}'|} \end{cases} \tag{14}$$

are the RMJ potentials and $\mu_{\alpha\beta} = m_\alpha m_\beta/(m_\alpha + m_\beta)$ is the reduced mass. This formulation allows to appreciate the Fokker–Planck structure of the Landau collisional operator.

By inserting equation (13) in (12), one gets:

$$\tilde{C}_{\alpha,\boldsymbol{m},\boldsymbol{k}}^{\mathrm{LAN}}(t) = \sum_\beta \frac{2\pi q_\alpha^2 q_\beta^2 \ln\Lambda}{m_\alpha v_{\mathrm{th}\alpha,0}^2}\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2,\boldsymbol{k}_2}\tilde{f}_{\alpha,\boldsymbol{m}_2,\boldsymbol{k}_2} \times\left[\frac{(\hat{a}_i\hat{a}_j g_{\beta,\boldsymbol{m}_1,\boldsymbol{k}-\boldsymbol{k}_2})I_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2}}{2m_\alpha v_{\mathrm{th}\alpha,0}^2} + - \frac{(\hat{a}_i h_{\beta,\boldsymbol{m}_1,\boldsymbol{k}-\boldsymbol{k}_2})J_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2}}{\mu_{\alpha\beta}}\right], \tag{15}$$

where:

$$I_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2} = \int \mathrm{d}^3\zeta_\alpha\,\boldsymbol{\Psi}^{\boldsymbol{m}}[(\hat{a}_i^\dagger\hat{a}_j^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_1})\boldsymbol{\Psi}_{\boldsymbol{m}_2} + + (\hat{a}_i^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_1})(\hat{a}_j^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_2}) + (\hat{a}_j^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_1})(\hat{a}_i^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_2}) + + \boldsymbol{\Psi}_{\boldsymbol{m}_1}(\hat{a}_i^\dagger\hat{a}_j^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_2})] \tag{16}$$

$$J_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2} = \int \mathrm{d}^3\zeta_\alpha\,\boldsymbol{\Psi}^{\boldsymbol{m}}[(\hat{a}_i^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_1})\boldsymbol{\Psi}_{\boldsymbol{m}_2} + + \boldsymbol{\Psi}_{\boldsymbol{m}_1}(\hat{a}_i^\dagger\boldsymbol{\Psi}_{\boldsymbol{m}_2})]. \tag{17}$$

The last two integrals contain the product of three Hermite functions and they are non-null in the case of even summation of the three involved Hermite coefficients [76]. In the asymptotic regime of large m ($m \sim m \pm 1$), the two integrals have the following dependence on the Hermite coefficient m: $I_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2} \sim m$ and $J_{\boldsymbol{m},\boldsymbol{m}_1,\boldsymbol{m}_2} \sim \sqrt{m}$.

By looking at equation (15), one easily realizes that nonlinearities of the Landau operator (i.e. the form of its Fokker–Planck coefficients) explicitly depend on the velocity coordinates. Therefore, equation (15) exhibits the convolutions over the Hermite coefficients. The Landau operator is hence non-local in the Hermite space: for a given Hermite coefficient $\boldsymbol{m}$, the Landau operator affects also the other Hermite modes. This represents a peculiar characteristic of the Landau operator, which is lost in other simplified operators, such as the Lenard–Bernstein or the Dougherty operator. In other words, the local coupling of Hermite modes, typical of collisionless systems (equation (10)), is drastically modified into a global coupling when introducing the Landau operator.

3.2. Collisions modeled through the Dougherty operator

Here we shift our focus on the case of the Dougherty operator, whose expression is:

$$C_\alpha^{\mathrm{DG}}(\boldsymbol{r}, \boldsymbol{v}, t) = \sum_\beta \nu_{\alpha\beta}(\boldsymbol{r}, t)\frac{\partial}{\partial v_i}\left[\frac{k_B T_{\alpha\beta}(\boldsymbol{r}, t)}{m_\alpha} \times \frac{\partial f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial v_i} + (v_i - u_{\alpha\beta,i}(\boldsymbol{r}, t))f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)\right]. \tag{18}$$

The collisional frequency $\nu_{\alpha\beta}(\boldsymbol{r}, t)$, the generalized speed $\boldsymbol{u}_{\alpha\beta}(\boldsymbol{r}, t)$ and the generalized temperature $T_{\alpha\beta}(\boldsymbol{r}, t)$ in equation (18) are obtained by adopting a 'simple' Fokker–Planck structure for the Dougherty operator and by expanding the Landau operator around an equilibrium distribution function f_0. Finally, by comparing the energy and momentum transfer equations obtained for the Landau and the Dougherty operators, it is possible to set the proper values for the parameters. (See [71] for a more detailed discussion.)

We notice that the Dougherty operator nonlinearities are intrinsically different from the Landau operator ones, since the Fokker–Planck coefficients of the Dougherty operator do not explicitly depend on the velocity coordinates. Therefore, contrary to the Landau operator case, we do not here expect to recover convolutions in the Hermite space. Indeed, by decomposing the Dougherty operator, one gets:

$$\tilde{C}^{\mathrm{DG}}_{\alpha,\boldsymbol{m},\boldsymbol{k}}(t) = \sum_\beta \sum_{\boldsymbol{k}_2,\boldsymbol{k}_3} \tilde{\nu}_{\alpha\beta,\boldsymbol{k}-\boldsymbol{k}_2-\boldsymbol{k}_3} \times \left[\left(\frac{\tilde{T}_{\alpha\beta,\boldsymbol{k}_3}}{T_{\alpha,0}} - \delta(\boldsymbol{k}_3)\right)(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{a}}) - (\tilde{\boldsymbol{u}}_{\alpha\beta,\boldsymbol{k}_3}\cdot\hat{\boldsymbol{a}})\right]\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}_2} + - \sum_\beta \sum_{\boldsymbol{k}_2} \tilde{\nu}_{\alpha\beta,\boldsymbol{k}-\boldsymbol{k}_2}(\hat{\boldsymbol{a}}^\dagger\cdot\hat{\boldsymbol{a}})\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}_2}, \tag{19}$$

where $\tilde{\nu}_{\alpha\beta,\boldsymbol{k}}$, $\tilde{T}_{\alpha\beta,\boldsymbol{k}}$ and $\tilde{\boldsymbol{u}}_{\alpha\beta,\boldsymbol{k}}$ are the Fourier coefficients of $\nu_{\alpha\beta}$, $\boldsymbol{u}_{\alpha\beta}$ and $T_{\alpha\beta}$ and the first term in equation (19) includes the spatial convolution due to the spatial dependence of $\boldsymbol{u}_{\alpha\beta}$ and $T_{\alpha\beta}$.

The behavior of the linear Lenard–Bernstein operator is recovered by linearizing the Dougherty operator. In this case, one has $T_{\alpha\beta} = T_{\alpha,0}$, $\boldsymbol{u}_{\alpha\beta} = \boldsymbol{u}_{\alpha0} = 0$ and $\nu_{\alpha\beta} = \nu_{\alpha\beta,0}$ and, hence $\tilde{T}_{\alpha\beta,\boldsymbol{k}} = T_{\alpha,0}\,\delta(\boldsymbol{k})$ and $\tilde{\boldsymbol{u}}_{\alpha\beta,\boldsymbol{k}} = 0$ and $\tilde{\nu}_{\alpha\beta,\boldsymbol{k}} = \nu_{\alpha\beta,0}\,\delta(\boldsymbol{k})$. Therefore, equation (19) reduces to:

$$\begin{aligned}\tilde{C}^{\mathrm{DG,lin}}_{\alpha,\boldsymbol{m},\boldsymbol{k}}(t) &= -\nu_{\alpha\beta,0}(\hat{\boldsymbol{a}}^\dagger\cdot\hat{\boldsymbol{a}})\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}\\ &= -\,\nu_{\alpha\beta,0}(m_x + m_y + m_z)\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}.\end{aligned} \tag{20}$$

As the Lenard–Bernstein operator, the linearized Dougherty operator is diagonal in the Fourier–Hermite space, with eigenvalue $-\nu_{\alpha\beta,0}(m_x + m_y + m_z)$ [20, 21]. We again emphasize that, as expected, the Dougherty operator acts locally in the Hermite space.

4. Implications for velocity-space enstrophy cascade

To analyze the effect of collisions in the Fourier–Hermite space, we obtain the equation for the Hermite spectrum $E_{\alpha,\boldsymbol{m},\boldsymbol{k}} = |\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}|^2$ [40] (the definition of the Hermite spectrum adopted here is slightly different from the one of [25, 41, 50]). By manipulating equation (10) coupled with equation (15) or, alternatively, with equation (19), the final result is:

$$\frac{\partial E_{\alpha,\boldsymbol{m},\boldsymbol{k}}}{\partial t} + \Lambda^{A,\alpha}_{\boldsymbol{m},\boldsymbol{k}} - \Lambda^{E,\alpha}_{\boldsymbol{m},\boldsymbol{k}} - \Lambda^{B,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = \Lambda^{\nu,\alpha}_{\boldsymbol{m},\boldsymbol{k}}, \tag{21}$$

where:

$$\begin{cases}\Lambda^{A,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = \mathrm{i}\, v_{\mathrm{th}\alpha,0}\,\boldsymbol{k}\cdot[\tilde{f}^*_{\alpha,\boldsymbol{m},\boldsymbol{k}}(\hat{\boldsymbol{a}}^\dagger + \hat{\boldsymbol{a}})\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}}] + \mathrm{c.c.}\\ \Lambda^{E,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = \frac{q_\alpha \tilde{f}^*_{\alpha,\boldsymbol{m},\boldsymbol{k}}}{m_\alpha v_{\mathrm{th}\alpha,0}}\sum_{\boldsymbol{k}_2}(\tilde{\boldsymbol{E}}_{\boldsymbol{k}-\boldsymbol{k}_2}\cdot\hat{\boldsymbol{a}})\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}} + \mathrm{c.c.}\\ \Lambda^{B,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = \frac{q_\alpha \tilde{f}^*_{\alpha,\boldsymbol{m},\boldsymbol{k}}}{m_\alpha c}\sum_{\boldsymbol{k}_2}(\hat{\boldsymbol{a}}^\dagger\times\tilde{\boldsymbol{B}}_{\boldsymbol{k}-\boldsymbol{k}_2})\cdot\hat{\boldsymbol{a}}\tilde{f}_{\alpha,\boldsymbol{m},\boldsymbol{k}} + \mathrm{c.c.}\\ \Lambda^{\nu,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = \tilde{f}^*_{\alpha,\boldsymbol{m},\boldsymbol{k}}\tilde{C}_{\alpha,\boldsymbol{m},\boldsymbol{k}} + \mathrm{c.c.}\end{cases} \tag{22}$$

and c.c. indicates the complex conjugate and $\tilde{C}_{\alpha,\boldsymbol{m},\boldsymbol{k}} = \tilde{C}^{\mathrm{LAN}}_{\alpha,\boldsymbol{m},\boldsymbol{k}}$ (equation (15)) or $\tilde{C}_{\alpha,\boldsymbol{m},\boldsymbol{k}} = \tilde{C}^{\mathrm{DG}}_{\alpha,\boldsymbol{m},\boldsymbol{k}}$ (equation (19)). Note that, in the linearized Dougherty operator case, $\Lambda^{\nu,\alpha}_{\boldsymbol{m},\boldsymbol{k}} = -\nu_{\alpha\beta,0}(m_x + m_y + m_z)E_{\alpha,\boldsymbol{m},\boldsymbol{k}}$.

By manipulating the operators given in equation (22) as $\Lambda^\alpha_{\boldsymbol{m},\boldsymbol{k}} \sim E_{\alpha,\boldsymbol{m},\boldsymbol{k}}/\tau^\alpha_{\boldsymbol{m},\boldsymbol{k}}$, one can obtain the characteristic times $\tau^\alpha_{\boldsymbol{m},\boldsymbol{k}}$ associated with each process, in the asymptotic regime of large m. We assume that each direction in both physical and velocity-spaces is equivalent: $m_x \simeq m_y \simeq m_z \simeq m$ and $k_x \simeq k_y \simeq k_z \simeq k$. Since we are interested in phenomenological scaling with m and k, we neglect the presence of convolutions in both Fourier and Hermite spaces. We also specialize in the case of a quasi-neutral plasma composed by protons and electrons with similar temperature ($T_p \simeq T_e$) and we focus on the proton dynamics ($\alpha = p$), thus simplifying the collisional operator structure by taking into account proton-proton collisions. This case is of particular interest for solar wind and inter-stellar medium applications. Bearing in mind that a factor $\sqrt{m}$ occurs each time an annihilation or

Table 1. Physical parameters, necessary to evaluate m_A^*, m_E^* and m_B^*, for the solar wind (first row), for the inter-stellar medium (center row) and for the hot-ion collisionless tokamak plasma (bottom row).

Plasmas	n_0	B_0	T_p	L_c	η	k
Slow solar wind	15 cm^{-3}	6nT	5×10^4 K	0.02 au	1	d_p^{-1}
Inter-stellar medium	3×10^{-3} cm^{-3}	10^{-6} G	10^6 K	1 kpc	0.5	ρ_p^{-1}
Hot-ion collisionless tokamak	4×10^{13} cm^{-3}	2T	2×10^3 keV	0.25 m	0.05	ρ_p^{-1}

creation operator is present, we obtain:

$$\begin{cases} \tau_{m,k}^{A,p} \sim 1/v_{\mathrm{thp},0} k\sqrt{m} \\ \tau_{m,k}^{E,p} \sim m_p v_{\mathrm{thp},0}/eE_k\sqrt{m} \\ \tau_{m,k}^{B,p} \sim m_p c/eB_k m \\ \tau_{m,k}^{\mathrm{LAN},p} \sim \tau_{pp}(1/m^2 - 1/m) \end{cases}, \tag{23}$$

where $\tau_{m,k}^{A,p}$, $\tau_{m,k}^{E,p}$, $\tau_{m,k}^{B,p}$ and $\tau_{m,k}^{\mathrm{LAN},p}$ are respectively the characteristic times associated with advection, electric field, magnetic field and collision terms in equation (21) and $\tau_{pp} = m_p^2 v_{\mathrm{thp},0}^3/2\pi e^4 n_p \ln\Lambda$ is the collisional time obtained by neglecting local velocity-space effects, i.e. the 'global' collisional time. Note that m_p is the proton mass and has not to be confused with the Hermite coefficient m.

The collisionless characteristic times derived here differ from the ones obtained in [50], since we do not average on the spatial domain and, hence, characteristic times also depend on the wavenumber k. The collisional characteristic time $\tau_{m,k}^{\mathrm{LAN},p}$ shows two different scaling with m: the diffusive (drag) term produces the scaling m^{-2} (m^{-1}). At large m, the fastest contribution is due to the diffusive term: $\tau_{m,k}^{\mathrm{LAN},p} \sim \tau_{pp}/m^2$. When focusing on the Dougherty operator case, in both linear and nonlinear regimes, the scaling is instead always m^{-1}.

It is possible to compare the characteristic times reported above to find Hermite coefficient m^* (at a given wavenumber k) when the plasma dynamics changes from a collisionless regime to a collisional one. We remark that, the theory in [50] is based on the conservation of enstrophy (or free energy) $\Omega = \int \mathrm{d}^3 v \delta f^2$ and this assumption breaks when introducing collisions. Indeed, by considering both the Landau or the Dougherty operators—which satisfy the H-theorem for the entropy growth—the enstrophy is not anymore preserved. In other words, m^* corresponds to the velocity-space scale at which the phenomenological theory described by Servidio *et al* [50] breaks its validity:

$$\begin{cases} m_A^* \simeq (v_{\mathrm{thp},0} k \tau_{pp})^{2/3} \\ m_E^* \simeq (\tau_{pp} eE_k/m_p v_{\mathrm{thp},0})^{2/3}, \\ m_B^* \simeq (\tau_{pp} eB_k/m_p c) \end{cases} \tag{24}$$

where m_A^*, m_E^* and m_B^* are respectively associated to the advection, electric field and magnetic field terms in equation (21), i.e. are respectively significant if the advection, electric field or magnetic field terms are dominant in the left-hand side of equation (21). In general, m^* decreases if the collisional characteristic time τ_{pp} decreases (i.e. collisions are faster), while it increases if the corresponding collisionless dominant term becomes more important (i.e. E_k, B_k or k larger). To appreciate the role of spatial fluctuations, we further simplify expressions in equation (24) by assuming a fully-developed turbulent scenario with a Kolmogorov scaling for both velocity and magnetic field fluctuations: $u_k = u_c(k/k_c)^{-1/3}$ and $B_k = B_c(k/k_c)^{-1/3}$, being $u_c = u(k_c)$ and $B_c = B(k_c)$ the fluctuation amplitudes at the correlation scale of the turbulence $L_c = 1/k_c$. The MHD scaling $E_k = u_k B_0/c$ is also assumed for the electric field fluctuations where B_0 is the background magnetic field. This choice is motivated by the fact that natural and laboratory plasmas are often strongly turbulent. Within these assumptions, it is easy to get:

$$\begin{cases} m_A^* \simeq (\sqrt{\beta_p}\ \Omega_{cp}\tau_{pp}\ kd_p)^{2/3} \\ m_E^* \simeq \left(\Omega_{cp}\tau_{pp}\ \frac{1}{\sqrt{\beta_p}}\frac{u_c}{c_A}\left(\frac{k}{k_c}\right)^{-1/3}\right)^{2/3} \\ m_B^* \simeq \left(\Omega_{cp}\tau_{pp}\ \frac{B_c}{B_0}\left(\frac{k}{k_c}\right)^{-1/3}\right) \end{cases}, \tag{25}$$

where $\Omega_{cp} = eB_0/m_p c$ is the proton cyclotron frequency, $d_p = c_A/\Omega_{cp}$ is the proton skin depth, $c_A = B_0/\sqrt{4\pi n_0 m_p}$ is the Alfvén speed, $\beta_p = 2v_{\mathrm{thp}}^2/c_A^2$ is the proton beta parameter and n_0 is the background density. The Hermite coefficients m_A^*, m_E^* and m_B^* depend on (i) the global collisional time normalized to the cyclotron time $\Omega_{cp}\tau_{pp}$, (ii) the proton beta β_p, (iii) the strength of turbulence $\eta \sim B_c/B_0 \sim u_c/c_A$ and (iv) the wavenumber k at which fluctuations are evaluated. Both m_E^* and m_B^* decrease as $k \gg k_c$: as turbulence produces smaller scale fluctuations the characteristic Hermite coefficients, relevant for turning on collisions, gets shorter. This last aspect represents, from a different point of view, the collisionality enhancement due to the presence of fine velocity scale structures, mainly produced by turbulent fluctuations that perturb the particle distribution function [12, 13].

We conclude the paper by calculating m_A^*, m_E^* and m_B^* for three typical weakly-collisional plasmas: the slow solar wind, the inter-stellar medium and the hot-ion collisionless tokamak plasma, which often shows a turbulent dynamics stirred by ITG. Relevant parameters for the calculation are extrapolated by [77–79, 2] for the solar wind; by [80] for the inter-stellar medium and by [81, 82] for the hot-ion collisionless tokamak plasma and are listed in table 1. In each case, the wavenumber k is close to the proton inertial scales (being $\rho_p = v_{\mathrm{thp}}/\Omega_{cp}$ the proton gyroradius).

Results are reported in table 2. In each system, the resulting Hermite coefficients useful to turn on collisions are quite large. This implies that: (i) the theory developed in [50]

Table 2. Hermite coefficients m_A^*, m_E^* and m_B^* for the solar wind (first row), for the inter-stellar medium (center row) and for the hot-ion collisionless plasma (bottom row).

Plasmas	m_A^*	m_E^*	m_B^*
Slow solar wind	1483	165	1800
Inter-stellar medium	10^6	325	19 000
Hot-ion collisionless tokamak	384	471	960

is valid up to these large Hermite coefficients; (ii) the presence of smaller scale spatial fluctuations may reduce m^*; (iii) potential failures of the collisionless assumption, induced by collisionality enhancement due to fine velocity-space structures, occur whether the distribution function exhibits structures such that their associated Hermite coefficient is $m \sim 200$. Recovering distribution function with such highly structured velocity-space perturbations is not nowadays possible due to velocity-space resolution limitations present in both spacecraft instruments and numerical simulations.

5. Conclusions

In this paper, the collisional Vlasov–Maxwell system of equations has been decomposed in the Fourier–Hermite space. This approach is extremely useful to describe fluctuations in velocity-space and the coupling of turbulent fluctuations in both physical and velocity-space. A compact notation, in terms of annihilation and creation operators, has been introduced. By modeling collisions through the Landau and the Dougherty (both nonlinear and linearized) operators, we have also decomposed the collisional operator in the Fourier–Hermite space. The features of the operators have been compared in detail. Finally we have shown that, by obtaining the equation for the Hermite spectrum $E_{\alpha,\boldsymbol{m},\boldsymbol{k}}$, it is possible to derive the scaling of each term in the collisional Vlasov equation. The characteristic times associated with each part of the Vlasov–Landau equation have been derived. These times are local in the Fourier–Hermite space and they can give insights on how fluctuations in both physical and velocity-spaces locally affect the relative importance of each term in the collisional Vlasov equation. The Hermite coefficients m^*, corresponding to the balance of the collisional time with the collisionless ones, have been derived separately for the advection, electric field and magnetic field terms. Under some assumptions, it has been possible to write simple expressions for m^*, that have been evaluated for three natural and laboratory plasmas: the slow solar wind, the inter-stellar medium and the hot-ion collisionless tokamak plasma. The resulting Hermite coefficients m^*, which correspond to the transition from a collisionless to a collisional regime in the plasma dynamics, are quite large for the considered systems. It is worth to note that the introduction of a collisional operator breaks the enstrophy conservation at small velocity scales, a principle which is analogous to the Kolmogorov cascade of energy for fluids.

Acknowledgments

OP and SS were partly supported by the International Space Science Institute (ISSI) in the framework of International Team 405 entitled Current Sheets, Turbulence, Structures and Particle Acceleration in the Heliosphere. EC was partially supported by NWO–Vidi grant 639.072.716. OP sincerely thanks Professor L Sorriso-Valvo for the enjoyable and fruitful discussions on these topics.

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